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Time Consistency, Temporal Resolution Indifference and the Separation of Time and Risk - Online Appendix

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APPENDIX A: HARA PREFERENCES

It would clearly be desirable to weaken the restriction that time and risk preferences must be homothetic. In fact, our TC result extends to the full class of HARA time and risk preferences if ICER holds and one of the available assets is risk free. The quasihomothetic members of the HARA class other than the CRRA case can be viewed as being homothetic to translated origins (see Pollak (1971)). The risk free asset assumption is crucial in dealing with the translations. Therefore, we assume that there exists a one period risk free asset at each date-event or a linear combination of the risky assets with the weight ω that yields the same payoff on each branch.

Assumption RF For each s^t, $t = 1, \ldots, T-1$, there exists an $\omega(s^t)$, where $\omega = \omega_1, ..., \omega_J$, such that

$$\sum_j \omega_j \left(s^t\right) R_j \left(s^{t+1}\right) = 1 \;\; \forall s^{t+1} \succ s^t.$$

Note that this assumption is automatically satisfied when markets are complete.

Quasihomothetic preferences also generate linear Engel curves, but which may not go through the origin. DOCE preferences will be homothetic (quasihomothetic) if and only if its building block utilities are homothetic (quasihomothetic). The quasihomothetic time and risk preference utilities can take the form

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2}, \tag{A.1}$$

where $(\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b))$,

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \;\; (\kappa_1, \kappa_2 > 0) \tag{A.2}$$

and

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \;\; (\delta_1, \delta_2 < -1, b > c > 0). \tag{A.3}$$

For the NM indices in (A.1), (A.2) and (A.3), respectively, the risk preferences exhibit decreasing, constant and increasing absolute risk aversion. This collection of NM indices is typically referred to as the HARA class. The corresponding certainty utilities are frequently referred to as the Modified Bergson family.[1] For the popular DARA (decreasing absolute risk aversion) case (A.1), it is standard to interpret $b > 0$ as a certain subsistence requirement.[2]

We next consider the case of DOCE preferences corresponding to the HARA (and modified Bergson) utilities (A.1) - (A.3). The key insight is that if Assumption RF holds and U and V, respectively, take the shifted CES and CRRA forms with the same shift parameters, the problem can simply be viewed as consumers having homothetic preferences for consumption in excess of their subsistence consumption requirements and a modified version of the results in Theorem 1 apply. Although Assumption ICER continues to play a key role for the DARA and IARA members of the HARA class, for the CARA member one can make the weaker assumption that the risk free rate R_{ft} is "non-stochastic" or constant across branches. It is striking that for this case, no restriction need be made on risky asset returns.[3] Then, we have the following result for DOCE consumption and asset demands to be time consistent.

THEOREM A.1. *Suppose the consumer solves the consumption-portfolio problem (10) - (13) and Assumption RF holds.*

(i) *Assumption ICER with $\widetilde{\mathbf{n}}(s^t)$ as defined in (30) holds. Then DOCE demands are time consistent if the DOCE time and risk preference utilities take the forms*

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2},$$

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[1]See Pollak (1971) for a description of the Modified Bergson class.

[2]For the DARA case we can have $b < 0$, but then the subsistence interpretation does not make sense (see Pollak (1970, p. 748)). For the IARA (increasing absolute risk aversion) case (A.3), b can be interpreted as a bliss point.

[3]The dual assumptions of negative exponential Expected Utility preferences and non-stochastic interest rates are sometimes made in models of asset pricing under asymmetric information and in microstructure analyses (e.g., Wang (1993) and Wang (1994)).

where $(\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b))$*, or*

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0);$$

(ii) *The risk free interest rate R_{ft} is non-stochastic. Then DOCE demands are time consistent if the DOCE time and risk preference utilities take the forms*

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \textit{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0).$$

PROOF OF THEOREM A.1. For (i) note that the maximization problem of an individual with utilities given by (A.1) is identical to the maximization problem of an individual who has utilities given by (6) except that she needs to purchase b/R_f units of the risk free asset at each s^t, $t < T$, to fund her subsistence requirement b. For the case (A.3), one can apply a similar argument. Since this in turn is equivalent to a problem where the individual's utility is homothetic and her income is appropriately adjusted, ICER remains necessary and sufficient for TC given the homothetic utility function.

For (ii) consider the first order conditions for optimal choice at some date-event s^τ of assets at some future $t > \tau$ for the risk free asset

$$V'(c(s^t))\left(u \circ V^{-1}\right)'\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right) =$$

$$\beta(u \circ V^{-1})'\left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right)\sum_{s^{t+1} \succ s^\tau} R_{ft+1}\pi(s^{t+1}|s^\tau)V'\left(c\left(s^{t+1}\right)\right).$$

Adding over all $s^t \succ s^\tau$ weighted by $\pi(s^t|s^\tau)$, taking into account that $V'(c) = -V(c)/\kappa_2$ we obtain

$$\left(u \circ V^{-1}\right)\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right) =$$

$$\beta(u \circ V^{-1})\left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right)R_{ft+1}.$$

Since this holds for any s^τ, it follows that the first order conditions for the optimal choice do not change with τ and hence the choice satisfies time consistency. □

The proof of Theorem A.1(i) follows essentially from the CES/CRRA utility case in Theorem 1. Due to the presence of a risk free asset, the consumer's maximization problem can be viewed as that of another consumer who owns enough of the risk free asset to pay b in each period $t = 1, ..., T-1$ and maximizes the CES/CRRA utility.

One may wonder why for the CARA case, Theorem A.1(ii), no restriction on risky asset returns such as ICER is assumed as in Theorems 1 and A.1(i). The explanation follows immediately from the well-known property of CARA risk preferences that the demand for the risky asset n is independent of investment. As a result, the certainty equivalent portfolio return $\widehat{R}_{pt}$ equals the risk free rate R_{ft}. For any period t, an increase in $(I_{t-1} - c_{t-1})$ results only in an increase in the holdings of the risk free asset n_{ft-1} and an incremental increase in the portfolio return equal to R_{ft}.[4]

REMARK A.1. It should be noted that (i) for both Theorems 1 and A.1, the additive time preference U, eqn. (1), can have an arbitrary period 1 utility $u_1(c_1)$ which satisfies $u_1' > 0$ and $u_1'' < 0$ and (ii) for Theorem A.1, the cases covered for risk preferences include the full HARA class.

We next show that DOCE and KP demands are the same for the building blocks in Theorem A.1.

PROPOSITION A.1. *Suppose the consumer solves the consumption-portfolio problem (10) - (13) and Assumption RF holds. For DOCE preferences,*

[4]For a more complete discussion of this phenomenon in the simple two period setting, see Selden and Wei (2024).

(i) *if Assumption ICER holds and we assume the time and risk preference building blocks (A.1), then the optimal demands can also be rationalized by KP preferences, where*

$$U(c_t, x) = -\frac{\left((c_t - b)^{-\delta_1} + \beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \textit{and} \quad V_T(x) = -\frac{(x-b)^{-\delta_2}}{\delta_2};$$

(ii) *if the risk free rate R_{ft} is non-stochastic and we assume the time and risk preference building blocks (A.2), then the optimal demands can also be rationalized by KP preferences, where*

$$U(c_t, x) = -\frac{\left(\exp(-\kappa_1 c_t) + \beta(-\kappa_2 x)^{\frac{\kappa_1}{\kappa_2}}\right)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2} \quad \textit{and} \quad V_T(x) = -\frac{\exp(-\kappa_2 x)}{\kappa_2};$$

(iii) *if Assumption ICER holds and we assume the time and risk preference building blocks (A.3), then the optimal demands can also be rationalized by KP preferences, where*

$$U(c_t, x) = \frac{\left((b - c_t)^{-\delta_1} + \beta(\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \textit{and} \quad V_T(x) = \frac{(b-x)^{-\delta_2}}{\delta_2}.$$

PROOF OF PROPOSITION A.1. Part (i) follows from exactly the same argument as in the proof of Theorem A.1. For both preference specifications we can rewrite the optimization problem as maximizing homothetic utility subject to an adjusted income and Proposition 2 then implies equivalence.

To prove (ii), as in the Proof of Theorem A.1, it can be verified that with the non-stochastic R_{ft}, the first order conditions of KP and DOCE will converge. Therefore, the two preferences generate the same optimal demands. Then similar to the proof of Proposition 2, one can show that the utility functions of the KP and DOCE preferences corresponding to these optimal demands are identical. This completes the proof. The intuition of this result can be also understood by noticing that (see Pollak (1971))

$$\lim_{\delta \to \infty} \left(1 + \frac{\kappa}{\delta} x\right)^{-\delta} = -\exp(\kappa x).$$

Since the CARA case can be viewed as the limit of the DARA case with infinite risk aversion and $\widehat{R}_{pt}$ will converge to R_{ft} for the infinite risk aversion, the results for the CARA case directly follow from the DARA case.

For the IARA case in part (iii), one can apply a similar argument as part (i) since this is equivalent to a problem where the individual's utility is homothetic and her income is appropriately adjusted. □

REMARK A.2. For the CARA case considered in Proposition A.1(ii), the consumer receives no new information about the risk free rate of interest R_{ft} with the passage of time. The fact that risky asset returns may change with the passage of time does not matter, since as indicated above $\widehat{R}_{pt} = R_{ft}$ and any change in beginning of period income $(I_{t-1} - c_{t-1})$ only affects risk free asset demand.

Finally, it should be noted that case (ii) in Proposition A.1 can be viewed as a special case of Hansen and Sargent (1995) risk-sensitive preferences, where the first term of $U(c_t, x)$ is specialized to the negative exponential form. When Assumptions ICER and RF hold, this special representation of risk-sensitive utility and the corresponding DOCE utility based on the same building blocks become identical on the analogue of $\mathcal{I}$ and generate the same optimal consumption and asset demands.

APPENDIX B: SAVING BEHAVIOR IN THE CONSUMPTION-PORTFOLIO PROBLEM

Use λ to denote the mean preserving spread parameter and increasing λ suggests that the spread increases. First we prove that when $\lambda_2 > \lambda_1$, we have $\widehat{R}_p(\lambda_1) > \widehat{R}_p(\lambda_2)$. Note that for any given (n, n_f), since the mean preserving spread is a special form of second order stochastic dominance, we have

$$V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)n + \xi_f n_f\right)\right] \geq V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_2)n + \xi_f n_f\right)\right].$$

Denoting the optimal asset demands associated with $\widetilde{\xi}(\lambda_1)$ as $\left(n^*, n_f^*\right)$ and associated with $\widetilde{\xi}(\lambda_2)$ as $\left(n^{**}, n_f^{**}\right)$, then

$$V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)n^* + \xi_f n_f^*\right)\right] > V^{-1}E\left[V\left(\widetilde{\xi}(\lambda_1)n^{**} + \xi_f n_f^{**}\right)\right]$$

$$\geq V^{-1} E\left[V\left(\widetilde{\xi}\left(\lambda_2\right) n^{**}+\xi_f n_f^{**}\right)\right].$$

Since

$$\widehat{R}_p=\frac{\widehat{c}_2}{I-c_1}=\frac{\left(E\left[\left(\widetilde{\xi} n+\xi_f n_f\right)^{-\delta_2}\right]\right)^{-\frac{1}{\delta_2}}}{I-c_1},$$

we have $\widehat{R}_p\left(\lambda_1\right)>\widehat{R}_p\left(\lambda_2\right)$, or

$$\frac{\partial \widehat{R}_p}{\partial \lambda}<0.$$

Under Assumption ICER, KP and DOCE preferences generate the same demands and for the three period case, it can be verified that

$$c_1=\frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1} \widehat{R}_p^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}},$$

which is a decreasing function of $\widehat{R}_p^{-\delta_1/(1+\delta_1)}$ and hence the saving $s_1=I-c_1$ is an increasing function of $\widehat{R}_p^{-\delta_1/(1+\delta_1)}$. Similarly, for the four period case,

$$c_1=\frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}\right)^{\delta_1+1}\right)^{\delta_1+1} \widehat{R}_p^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}}.$$

which is also a decreasing function of $\widehat{R}_p^{-\delta_1/(1+\delta_1)}$ and s_1 is an increasing function of $\widehat{R}_p^{-\delta_1/(1+\delta_1)}$. Based on induction, the result that s_1 is a increasing function of $\widehat{R}_p^{-\delta_1/(1+\delta_1)}$ can be extended to the T period case. Since

$$\frac{\partial \widehat{R}_p^{-\frac{\delta_1}{1+\delta_1}}}{\partial \lambda} \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0,$$

we have

$$\frac{\partial s_1}{\partial \lambda} \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0. \tag{B.1}$$

REMARK B.1. Suppose that ICER holds and one makes the admittedly strong assumption that the CRRA risk preference parameter $\delta_2 = 0$. Then following eqn. (B.1), the DOCE consumer will increase saving if the time preference parameter $\delta_1 > 0$. However, a EZW consumer who exhibits a preference for early resolution of risk (corresponding to $0 = \delta_2 > \delta_1$) will fail to increase saving. Clearly in this special case, the EZW consumer's temporal resolution preferences are confounding the roles of the time and risk preferences in explaining when the consumer will increase saving in response to an increase in risk.

APPENDIX C: NUMERICAL SIMULATION METHODOLOGY

We need to solve numerically for sophisticated choice. Resolute choice can, in principle be derived analytically via CES demand functions but it is simple to also solve for this numerically. For simplicity we assume that markets are incomplete. Without loss of generality we can restrict ourselves to a full set of Arrow securities being traded at any date-event. This simplifies the notation in the description of the computational method.

C.1 *Sophisticated choice*

Sophisticated choice can be solved by backward induction as follows.

For $\tau < t$, define $\widehat{c}^{\tau}_{ts}(a)$ to be the condition certainty equivalent at time t, given the next period shock is s as a function of holdings in Arrow security s at time τ. Formally, this depends on s^τ and we have

$$\widehat{c}^{s^\tau}_{ts}(a) = v^{-1} \sum_{s^t \succeq (s^\tau, s)} \pi(s^t | s^\tau) v(c(s^t; a)).$$

At each τ, we then have the following maximization problem

$$\max_{(a_1 \dots a_S)} u(w - q \cdot a) + \sum_{t=\tau+1}^{T} u \left(v^{-1} (\sum_{s=1}^{S} \pi(s|\bar{s}) v(\widehat{c}^{\tau}_{ts}(a_s))) \right). \tag{C.1}$$

This may appear complicated, but the key simplification comes from the following lemma.

LEMMA C.1. *Suppose $v(.)$ and $u(.)$ are homothetic. Then each $\widehat{c}^{\tau}_{ts}(.)$ is linear, i.e.,*

$$\widehat{c}^{\tau}_{ts}(a) = \gamma^{\tau}_{ts} a \text{ for some } \gamma^{\tau}_{ts} > 0.$$

Under homothetic utility, the functions $\widehat{c}^{\tau}_{ts}$ can be computed by backward induction. The computation also directly proves the lemma by backward induction. Obviously, in the last period, the agent consumes everything and $\widehat{c}^{T-1}_{Ts}(a) = a$. When all future certainty equivalents are linear, it is easy to verify that Arrow security holdings that solve (C.1) are a linear function of beginning-of-period wealth, w (this will also become clear below). For $T-1$ we have that in all possible shocks $s = 1, \ldots, S$, the holding of Arrow security s' is given by $a(s'|s) = \alpha^{s}_{s',T-1} w$. Therefore, we have that the last period certainty equivalent at $T-2$ can be expressed as $\widehat{c}^{T-2}_{Ts}(a) = v^{-1}\left(\sum_{s'} \pi(s'|s) v(\alpha^{s}_{s',T-1} a)\right)$, and hence $\gamma^{T-2}_{T,s} = v^{-1}\left(\sum_{s'} \pi(s'|s) v(\alpha^{s}_{s',T-1})\right)$. It is easy to see that the same argument can be made for $T-3$...

The following first-order conditions characterize optimal choice. Given shock $\bar{s}$, for all $s = 1, \ldots, S$

$$\begin{aligned} & q(s|\bar{s}) u'(w - q \cdot a) = \\ & \sum_{t=\tau+1}^{T} \beta^{t-\tau} \pi(s|\bar{s}) v'(\gamma^{\tau}_{ts} a_s) \gamma^{\tau}_{ts} u' \left(v^{-1} (\sum_{s'=1}^{S} \pi(s'|\bar{s}) v(\gamma^{\tau}_{ts'} a_{s'})) \right) \\ & v^{-1\prime} \left(\sum_{s'=1}^{S} \pi(s'|\bar{s}) v(\gamma^{\tau}_{ts'} a_{s'}) \right). \end{aligned}$$

Using the inverse function theorem, and the notation

$$v^{-1} \left(\sum_{s'=1}^{S} \pi(s'|\bar{s}) v(\gamma^{\tau}_{ts'} a_{s'}) \right) = \gamma^{\tau-1}_{t\bar{s}} w,$$

we obtain

$$\begin{aligned} & q(s|\bar{s}) u'(w - q \cdot a) = \\ & \sum_{t=\tau+1}^{T} \beta^{t-\tau} \pi(s|\bar{s}) v'(\gamma^{\tau}_{ts} a_s) \gamma^{\tau}_{ts} \frac{u'\left(\gamma^{\tau-1}_{t\bar{s}} w\right)}{v'\left(\gamma^{\tau-1}_{t\bar{s}} w\right)}. \end{aligned}$$

It is trivial to use standard root-finding methods to solve numerically for a.

C.2 *Resolute choice*

It is useful to construct a price for each $\widehat{c}_t$, $t=0,\ldots,T$ and then simply solve

$$\max \sum_{t=0}^{T} \beta^t u(\widehat{c}_t) \;\; S.T. \; \sum_{t=0}^{T} P(\widehat{c}_t)\widehat{c}_t = I. \tag{C.2}$$

The first order conditions are necessary and sufficient.

Defining $p(s^t)$ to be the date zero price of consumption at s^t, these prices can be defined as follows.

$$P(\widehat{c}_t) = \min_{(c(s^t))} \sum_{s^t} p(s^t)c(s^t) \;\; S.T. \;\; v^{-1}\left(\sum_{s^t} \pi(s^t)v(c(s^t))\right) \geq \widehat{c}_t.$$

The first order conditions imply that for any s^t, $\widehat{s^t}$,

$$c(s^t) = v'^{-1}\left(\frac{\frac{p(s^t)}{\pi(s^t)}}{\frac{p(\widehat{s^t})}{\pi(\widehat{s^t})}}\right) c(\widehat{s^t}).$$

The prices can be computed recursively as follows. Define the stochastic discount factor as

$$\rho(s'|s) = \frac{q(s'|s)}{\pi(s'|s)}$$

and let $P^t_s(\widehat{c}_t) = 1$ and recursively for $\tau < t$

$$P^\tau_s(\widehat{c}_t) = \frac{\sum_{s'} q(s''^{-1}\left(\frac{\rho(s'|s)}{\rho(1|s)}\right) P^{\tau+1}_{s'}(\widehat{c}_t)}{v^{-1}\left(\sum_{s'} \pi(s''^{-1}\left(\frac{\rho(s'|s)}{\rho(1|s)}\right)\right)}.$$

The first order conditions of (C.2) are necessary and sufficient and once the $P(\widehat{c}_t)$ are computed the problem becomes easy to solve.

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