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Can intergenerational equity be operationalized?: Correction

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We show with a counterexample that Theorem 1 of [Zame, William R. "Can intergenerational equity be operationalized?" Theoretical Economics 2.2 (2007): 187-202.] is wrong. The problem with Theorem 1 does not affect the remaining results of the paper, which form its core.

KEYWORDS. Intergenerational equity, infinite utility streams, Pareto.

JEL CLASSIFICATION. D60, D63.

1. INTRODUCTION

In the context of intergenerational equity,[1] Zame (2007) shows that a preference relation on infinite utility streams that is invariant under permuting finitely many indices and that satisfies the weak Pareto criterion gives rise to a nonmeasurable set of reals (Theorem 2). This implies, in combination with a consistency result known from set theory, that such a relation cannot be shown to exist if one weakens the axiom of choice to the axiom of dependent choice (Theorem 3).[2] This confirms a previous conjecture of Fleurbaey and Michel (2003). Furthermore, it is consistent that such preferences are not definable (Theorem 4).

As a preliminary result for proving the remaining theorems, Zame argues (Theorem 1 and, for a different domain, Theorem 1') that an irreflexive strict preference relation on infinite utility streams that is invariant under switching finitely many indices must be extremely indecisive in that the set of pairs of utility streams that cannot be strictly compared must have full outer measure. This is not so. We show that preferences on infinite utility streams can be quite decisive, even when we require them to satisfy the strong

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[1] See Pivato and Fleurbaey (2024) for a survey of intergenerational equity that also discusses different interpretations of preference relations on infinite utility streams.

[2] There are similar results by Lauwers (2010), Dubey (2011) and Dubey and Laguzzi (2023). Lauwers (2010) and Dubey (2011) show that the existence of such preferences implies the existence of a non-Ramsey set, and Dubey and Laguzzi (2023) show that the existence of such preferences implies the existence of a non-Baire set. By results of Shelah (1984), it is relatively consistent that dependent choice holds and all sets of reals are Baire sets, while the consistency of dependent choice and all sets of reals being measurable holds only relatively to an inaccessible cardinal. The consistency strength of having dependent choice and only Ramsey sets remains unknown.

Pareto criterion. For the preferences we construct, the set of pairs of utility streams that cannot be strictly compared (in our case, the graph of the indifference relation) is measurable and has measure zero. To "construct" such preferences, a tiny modification of an argument of Svensson (1980) suffices. We later discuss the mistake in the proof of Theorem 1 and explain why it does not affect its uses in the proofs of the remaining theorems.

2. Preliminaries and Counterexample

A binary relation $\geq$ on a set is a *partial order* if it is reflexive, transitive, and anti-symmetric ($x \geq y$ and $y \geq x$ imply $x = y$). A complete partial order is a *linear order*. A complete and transitive preference relation corresponds naturally to a linear order on the indifference classes.

For two sequences $x = (x_n)$ and $y = (y_n)$ in $\mathbb{R}^{\mathbb{N}}$, we write $x \geq y$ if $x_n \geq y_n$ for all $n \in \mathbb{N}$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_n > y_n$ for all $n \in \mathbb{N}$. For a generic (binary) relation $\succeq$, we denote its asymmetric part by $\succ$ and its symmetric part by $\sim$. A relation $\succeq$ on a subset of $\mathbb{R}^{\mathbb{N}}$ satisfies *strong Pareto* if $x > y$ implies $x \succ y$, and satisfies *weak Pareto* if $x \gg y$ implies $x \succ y$.

A *finite permutation* is a permutation (bijection) $\sigma : \mathbb{N} \to \mathbb{N}$ such that $n = \sigma(n)$ for all but finitely many $n \in \mathbb{N}$. We denote the set of finite permutations by $\mathbb{F}$. Under the operation of composition, $\mathbb{F}$ is a countable group. If $x \in \mathbb{R}^{\mathbb{N}}$ and $\sigma \in \mathbb{F}$, we write, slightly abusing notation, $\sigma(x)$ for the sequence $(x_{\sigma(n)})$. A relation $\succeq$ on $X = [0,1]^{\mathbb{N}}$ or $X = \{0,1\}^{\mathbb{N}}$ satisfies *intergenerational equity* if $x \succ y$ holds if and only if $\sigma(x) \succ \tau(y)$ for all $x, y \in X$ and $\sigma, \tau \in \mathbb{F}$. The following characterization of intergenerational equity for well-behaved preferences (already noted by Zame (2007)) is straightforward to prove.

Fact. *Let $\succeq$ be a complete and transitive relation on $X = [0,1]^{\mathbb{N}}$ or $X = \{0,1\}^{\mathbb{N}}$. Then $\succeq$ satisfies intergenerational equity if and only if $x \sim \sigma(x)$ holds for all $x \in X$ and $\sigma \in \mathbb{F}$.*

We endow $X = [0,1]^{\mathbb{N}}$ and $X = \{0,1\}^{\mathbb{N}}$ with their natural product topologies and Borel σ-algebras. Either space is Polish (separable and completely metrizable). We endow (the Borel sets of) $X = [0,1]^{\mathbb{N}}$ with the countable product of Lebesgue measure, and (the Borel sets of) $X = \{0,1\}^{\mathbb{N}}$ with the countable fair coin-flipping measure. In either case, we endow (the Borel sets of) $X \times X$ with the resulting product measures. These product measures are atomless, which in the present context is equivalent to no point having strictly positive mass. A property holds for *almost all* pairs in $X \times X$ if it holds outside a Borel set of measure zero. The outer measure of an arbitrary subset of $X \times X$ is the infimum of the measures of all Borel supersets. The inner measure of an arbitrary subset of $X \times X$ is the supremum of the measures of all Borel subsets.

Theorem 1. *There exist complete and transitive relations on $X = [0,1]^{\mathbb{N}}$ and $X = \{0,1\}^{\mathbb{N}}$ that satisfy strong Pareto, intergenerational equity, and such that almost all pairs in $X \times X$ are strictly comparable.*

PROOF. Write $x \equiv y$ if there exists a finite permutation σ such that $\sigma(x) = y$. The group structure on $\mathbb{F}$ implies that $\equiv$ is an equivalence relation. By Kechris (1995, Theorem 6.4), the graph of a Borel measurable function between two Polish spaces is measurable. By Fubini's theorem, it has measure zero under any product probability measure with atomless marginals. Consequently, the graph of $\equiv$ can be written as a countable union of measure zero sets as

$$\bigcup_{\sigma \in \mathbb{F}} \Big\{ \big(x, \sigma(x)\big) \mid x \in X \Big\}$$

and is, therefore, a measure zero set itself. We define a relation $\geqq$ on the partition into equivalence classes $X/\equiv$ such that for two equivalence classes, $[x]$ and $[y]$, we have $[x] \geqq [y]$ exactly when $x \geq \sigma(y)$ holds for some finite permutation σ. Then, $\geqq$ is easily seen to be a partial order. By the Szpilrajn extension theorem, Szpilrajn (1930), $\geqq$ extends to a linear order on $X/\equiv$. We now define the desired relation $\succeq$ on X so that $x \succeq y$ holds exactly when $[x] \geqq [y]$. The set of pairs that are not strictly comparable is then exactly the graph of $\equiv$, a set of measure zero. □

The "construction" we used in the proof of Theorem **??** is virtually identical to the one used by Svensson (1980). The only difference is that we use an extension from partial orders to linear orders, while Svensson uses an extension from preorders to complete preorders that preserves strict preferences. The latter kind of extension might create additional indifferences, the former does not.

3. DISCUSSION

Zame (2007) uses his Theorem 1 (and 1') to prove the remaining theorems in his paper. However, this does not cause any problems for the remaining theorems. In the proof of his Theorem 1, Zame starts with a strict relation $\succ$ on X that satisfies intergenerational equity and proves that the sets

$$L = \big\{(x,y) \in X \times X \mid x \succ y\big\} \text{ and } R = \big\{(x,y) \in X \times X \mid y \succ x\big\}$$

both have inner measure zero. He then concludes that the set

$$L \cup R = \big\{(x,y) \in X \times X \mid x \succ y \text{ or } y \succ x\big\}$$

has inner measure zero, too. This last step is not warranted; inner measure is superadditive but, in general, not subadditive. But the step is warranted when these two sets are measurable, as Zame assumes (for the sake of contradiction) when he uses his Theorem 1 (and 1') to prove the remaining theorems. These theorems are all correct.

Theorem 2 of Zame (2007) states that a complete and transitive preference relation that satisfies intergenerational equity and weak Pareto must have a nonmeasurable graph. If the indifference relation has a measurable graph, this implies that the graph of the strict preference relation must be nonmeasurable. The graph of the strict preference relation, the set L, is maximally nonmeasurable in our example: The difference between the inner and outer measure is 1. Indeed, the complement of L consists of R and the graph of the indifference relation I. Since I is measurable with measure 0, the inner measure of $I \cup R$ equals the inner measure of R. So L has outer measure 1. Theorem 1 of Zame (2007) suggests that the graph of an "ethical" preference relation is an unruly object because I is. In our example, L deserves the blame instead.

References

Dubey, Ram Sewak (2011), "Fleurbaey-Michel conjecture on equitable weak Paretian social welfare order." *Journal of Mathematical Economics*, 47(4-5), 434–439. [1]

Dubey, Ram Sewak and Giorgio Laguzzi (2023), "Social welfare relations and irregular sets." *Annals of Pure and Applied Logic*, 174(9), Paper No. 103302, 19. [1]

Fleurbaey, Marc and Philippe Michel (2003), "Intertemporal equity and the extension of the Ramsey criterion." *Journal of Mathematical Economics*, 39(7), 777–802. [1]

Kechris, Alexander S. (1995), *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. [3]

Lauwers, Luc (2010), "Ordering infinite utility streams comes at the cost of a non-Ramsey set." *Journal of Mathematical Economics*, 46(1), 32–37. [1]

Pivato, Marcus and Marc Fleurbaey (2024), "Intergenerational equity and infinite-population ethics: a survey." *Journal of Mathematical Economics*, 113, Paper No. 103021, 21. [1]

Shelah, Saharon (1984), "Can you take Solovay's inaccessible away?" *Israel Journal of Mathematics*, 48(1), 1–47. [1]

Svensson, Lars-Gunnar (1980), "Equity among generations." *Econometrica*, 48(5), 1251–1256. [2, 3]

Szpilrajn, Edward (1930), "Sur l'extension de l'ordre partiel." *Fundamenta Mathematicae*, 16, 386–389. [3]

Zame, William R. (2007), "Can intergenerational equity be operationalized?" *Theoretical Economics*, 2(2), 187–202. [1, 2, 3, 4]

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