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Time Consistency, Temporal Resolution Indifference and the Separation of Time and Risk

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For general choice spaces, standard dynamic preference models cannot simultaneously satisfy the properties of time consistency, the separation of time and risk preferences and the ability to accommodate an indifference to the timing of when risk is resolved. In the context of a consumption-portfolio choice problem often underlying asset pricing and macro models, we derive necessary and sufficient conditions such that all three properties are satisfied. We also show that quantitatively reasonable deviations from our sufficient conditions can result in surprisingly small deviations from time consistency holding.

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1. INTRODUCTION

For economic models where consumers solve dynamic optimization problems under risk, assumptions on preferences play a key role in the resulting solutions and their comparative statics. At the level of preferences, the following three properties are often mentioned as being desirable: (i) time consistency (TC), (ii) the ability to separate time and risk preferences (SEP) and (iii) the ability to accommodate temporal resolution of risk indifference (TRI). However, standard dynamic preference models cannot simultaneously satisfy these three properties. In this paper in the context of the dynamic consumption-portfolio problem, we provide necessary and sufficient conditions on preferences such that the three properties can be satisfied on a meaningful subset of the general choice space. Moreover, conditions on asset returns are given that ensure that the optimal demands based on these preferences always lie in this set. When all three properties are satisfied, one can unambiguously separate the specific roles of time and risk preferences in explaining asset demand and intertemporal consumption-saving behavior.

In deriving conditions such that TC, SEP and TRI hold, we focus on the DOCE (dynamic ordinal certainty equivalent) preference structure of Selden and Stux (1978). We also consider the better known model of KP (Kreps and Porteus (1978)). As the axiomatic framework for both DOCE and KP preferences is fully developed, our primary concern is with the respective demand implications of the two models. Unlike the DOCE preference model which assumes TRI, the KP preference model by design incorporates a pure psychic preference for early or late resolution of consumption risk (which should be distinguished from an early resolution of income risk which could be of significant planning value). It is well-known that temporal resolution preferences can confound the complete separation of the effects of time and risk preferences.[1] Moreover, in recent years some research has called into question the temporal resolution implications of KP preferences. Epstein, Farhi and Strzalecki (2014) argue that standard parameter assumptions in finance and macroeconomic applications of the EZW (Epstein and Zin (1989) and Weil (1990)) version of the KP preference model can imply that consumers would pay **implausibly** large timing premia for early resolution of consumption risk.[2] More directly, Meissner and Pfeiffer (2022) provide an experimental test of both (i) the existence and size of temporal resolution preferences of individuals and (ii) the validity of the EZW preference model. First, they directly estimate the timing premium and find that roughly 40% of their subjects exhibit TRI. Second, based on an independent elicitation of the subjects' time preference and risk preference parameters, they compute the theoretically predicted timing premia and find a negative correlation between the estimated and predicted timing premia. They conclude that the assumptions of the EZW preference model are inconsistent with their experimental results.

DOCE preferences are based on two independent building blocks: a one-period EU (expected utility) representation defined over distributions of consumption characterizing risk preferences and a multiperiod time preference utility over certain intertemporal consumption vectors. By construction, DOCE preferences always satisfy SEP and TRI, but do not satisfy TC for arbitrary choice spaces. Indeed, Epstein (1992, p. 19) observes that the Selden and Stux (1978) DOCE utility is appealing because of its "natural algorithm for computing utility

[1]Epstein and Zin (1989, p. 952) explicitly warn that "attitudes towards timing...seem intertwined with [time preference] substitutability and risk aversion." Weil (1990, p. 33) observes that it would be better if an alternative preference model would make it possible to disentangle a preference for early or late resolution from attitudes toward risk and intertemporal substitution. A similar desire for disentangling the three preference elements is expressed more recently by Barro and Ursua (2012, footnote 7). Also, see Strzalecki (2013).

[2]Epstein, Farhi and Strzalecki (2014) argue that for the Long Run Risk model of Bansal and Yaron (2004), consumers would give up approximately 30% of their lifetime consumption to have all risk resolved early even though this would have no instrumental value.

and [complete] separation between time and risk preferences", but is unsatisfactory because of its violation of TC. In this paper, we derive necessary and sufficient conditions such that TC also holds in a setting where the consumer solves a dynamic consumption-portfolio problem.

Suppose the consumer's underlying time and risk preference building blocks, respectively, take the CES (constant elasticity of substitution) and CRRA (constant relative risk aversion) homothetic forms. Then her demands will exhibit TC if and only if asset returns satisfy a property referred to as ICER (identical certainty equivalent return) where the asset portfolio certainty equivalent return in each time period is non-stochastic (i.e., it does not vary across nodes for a given time t).[3] The intuition for why DOCE preferences exhibit TC is that homotheticity and the ICER assumption permit the transformation of the choice over a multi-date-event branch consumption tree, such as in Figure 1 in the next section, into the choice over an equivalent single branch tree analogous to what the consumer confronts in a pure certainty time consistent setting.[4] At any given node of the dynamic consumption tree, since the assumed CRRA risk preferences imply one fund portfolio separation (Cass and Stiglitz (1970)), the ICER assumption ensures that the optimal asset mix will be the same for each branch. Thus, the consumer's portfolio composition will be the same irrespective of the state outcome at each node and she will have no reason to revise her plans as risk is resolved. Whereas in general DOCE preferences violate TC, our conditions are precisely what is required for TC to be satisfied.[5]

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[3]While the restriction that asset returns satisfy ICER is clearly a special case, the stronger assumption that asset returns are i.i.d. (identically and independently distributed) has been made in a number of important papers such as Levhari and Srinivasan (1969), Samuelson (1969) and Weil (1993). More recently, some research on rare disasters, such as Barro (2009), assumes i.i.d. distributions.

[4]It should be emphasized that TC depends not just on preferences but also on asset returns or prices. It is standard in discussions of time consistent preference models to impose restrictions solely on preferences and assume that the conditions hold for all prices (e.g., Blackorby, et al. (1973)). Our key property ICER can be viewed as effectively a restriction on asset prices and probabilities and thus TC holds only for a subset of prices. See further discussion on this point in Section 6.

[5]As shown in Supplemental Appendix A, the TC results obtained in this paper can be extended to the full class of HARA (hyperbolic absolute risk aversion) time and risk preferences.

Since KP and DOCE preferences can be constructed based on the same time and risk preference building block utilities, it is natural to wonder how the time consistent DOCE and KP demands relate to one another assuming asset returns satisfy ICER. Although the DOCE and KP utilities are not ordinally equivalent, under the assumptions guaranteeing that DOCE preferences satisfy TC, the DOCE and KP demand functions are identical. At first blush this seems quite surprising. However, this result can be understood once it is realized that on the restricted set of consumption trees corresponding to the consumption-portfolio problem, the DOCE and KP utilities actually become identical. This is true despite the fact that KP preferences typically exhibit a strict temporal resolution preference. The key to understanding this result is that the choice space over which the two preference relations agree excludes the set of early resolution trees that are essential to distinguishing a preference for early versus late resolution trees. As a result on the more general space of consumption trees, DOCE and KP consumers continue to diverge on their preferences over early and late resolution trees with KP individuals potentially being willing to pay implausibly large timing premia. This distinction implies that the DOCE preference model can be a useful alternative to the EZW special case of KP preferences even when the two models generate the same demands. Assume the DOCE and EZW preferences are based on the same time and risk preference building block utilities and that consumption growth is i.i.d. Then one can assume the DOCE preference model and generate exactly the same equilibrium asset returns as generated by the EZW preferences assumed in Epstein, Farhi and Strzalecki (2014), but without exhibiting the timing premia which the authors find objectionable. Moreover, this holds true for any combination of EIS and RRA (relative risk aversion) preference parameter values. It should be emphasized that, as indicated at the beginning of this section, our focus in this paper is on partial equilibrium consumption-portfolio demand analysis. However, as discussed briefly in our concluding comments in Section 6, the extension of our DOCE model to an analysis of equilibrium asset pricing when ICER does not hold and DOCE and EZW preferences diverge would seem to be a potentially interesting and important avenue for future research.

Although ICER is a strong condition, when assuming CES time and CRRA risk preference one can view it as the cost of ensuring that SEP holds and avoiding timing premia. When ICER does not hold, DOCE preferences no longer satisfy TC and the consumer can be viewed as exhibiting changing tastes as in the classic papers of Strotz (1956) and Pollak (1968). As a result, application of the standard resolute, naive and sophisticated solution techniques to the dynamic consumption-portfolio problem in general yield different consumption and asset demand functions. We identify restrictions on the consumer's preferences ensuring that departures from ICER lead to small differences between sophisticated and resolute demand which can be viewed as consistent with small welfare losses and TC holding approximately. We provide analytic results for infinitesimal change and numerical simulation results for discrete departures. We construct examples where the welfare costs of time inconsistency can become quite large, but argue that they may not be empirically relevant. Based on the same asset price process as is used by Melino and Yang (2003) and for which a calibration is available, we specifically estimate the welfare losses associated with different EIS and RRA combinations. With regard to the latter, as long as the $EIS \leq 0.5$ and $RRA = 10$, the welfare losses associated with resolute and sophisticated demand diverging are surprisingly small. For the case where $0.5 < EIS \leq 0.75$ and the $RRA = 10$, the welfare losses are larger but still only about 1%. In these cases, where the welfare losses are small and TC holds approximately, the assumption of CES-CRRA DOCE preferences allows one to realize the benefits of full separation of time and risk preferences and TRI being satisfied. For an $EIS = 1.5$ and $RRA = 10$, the welfare costs of time inconsistency are larger but still only about 4%.[6]

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[6] When ICER does not hold, we cannot directly compare the welfare losses of TC not holding exactly with the timing premium estimates in Epstein, Farhi and Strzalecki (2014). Moreover, Epstein, Farhi and Strzalecki (2014) consider very different stochastic processes than the simple 2 state process we examine in this paper. It is subject to further research to determine, when allowing for more general stochastic processes of prices in an equilibrium setting, under what conditions the timing premia are large for EZ and under what conditions welfare losses from time inconsistent DOCE preferences are large.

The rest of the paper is organized as follows. In the next section, we introduce notation, definitions and the consumer's optimization problem. Section 3 provides necessary and sufficient conditions for DOCE preferences to satisfy TC. When TC holds, the classic certainty Fisherian consumption-saving analysis can be applied to the consumption-portfolio setting. Section 4 compares consumption and asset demands for DOCE and KP preferences based on the same CES-CRRA time and risk preference building block utilities when asset returns satisfy ICER. In Section 5, we show that under given conditions, if one relaxes ICER, DOCE preferences can remain *almost* time consistent. Section 6 contains concluding comments. Proofs are given in the Appendix and additional material is provided in the online Supplemental Appendix.

2. Preliminaries

2.1 *Notation and Definitions*

We assume a risky, intertemporal setting where time is indexed by $t = 1, \ldots, T$. A consumer solves a dynamic consumption-portfolio problem (formally defined in the next subsection). The asset portfolio is comprised of risky assets and, in some cases, a risk free asset. Each asset has a maturity of one time period. At the beginning of each period $t < T$, conditional on asset return realizations and prior saving decisions which determine period t income, the consumer chooses optimal period t consumption and asset holdings and a plan for consumption and asset holdings for each future time period.[7] The risky asset return distribution determines the stochastic structure of the consumer's dynamic consumption possibilities and any budget feasible set of demands determines a consumption tree. This is illustrated in Figure 1 which can be viewed as corresponding to a feasible three period consumption plan where in the first two time periods the consumer can invest in a single risky and risk free asset. In periods 1 and 2, the risky asset has two possible payoffs with associated probabilities. This return distribution

[7]In the terminal period T, consumption corresponds to the return from the $T-1$ portfolio.

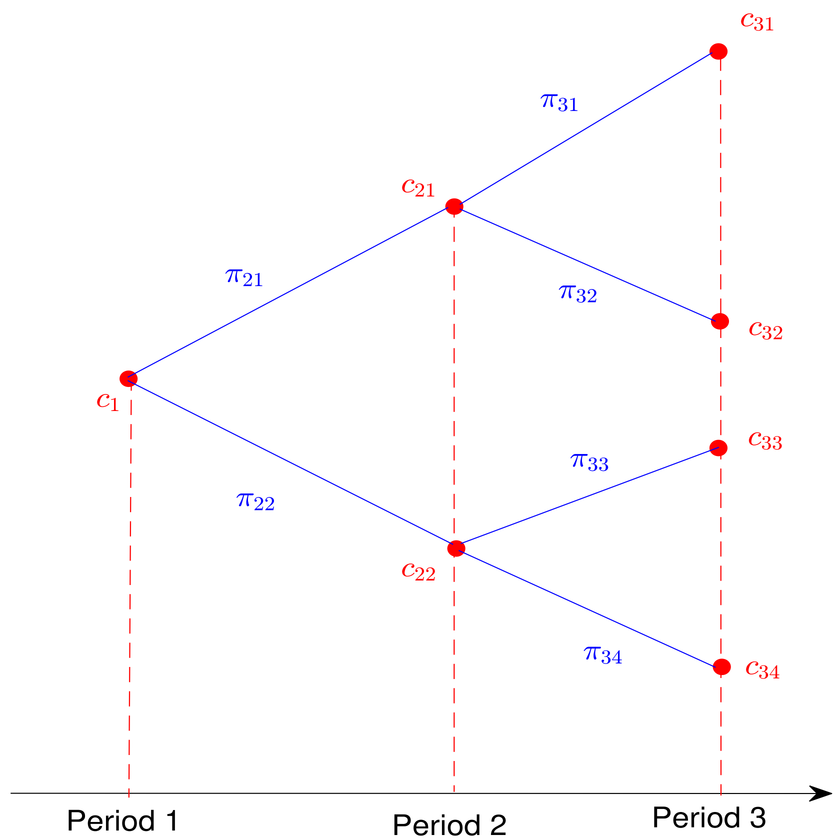

FIGURE 1. Three Period, Four Branch Consumption Tree

defines the four branch tree structure in Figure 1 as well as the branch probabilities. A node in this tree is the combination of a date t and the realization of the asset return distribution and is associated with a specific consumption value.

It should be emphasized that in solving the dynamic consumption-portfolio problem, the set of consumption trees over which the consumer is choosing has a fixed set of probabilities and branches specified by the assumed risky asset payoff distribution. Given that the results we prove in this paper hold for an arbitrary number of time periods and asset return realizations, it is necessary to introduce a more general notation than used for the simple tree in Figure 1 to identify branches and nodes of the tree. Following the standard practice in the dynamic asset pricing literature (e.g., Duffie (2001, Chapter 3)), different realizations of the asset returns are referred to as economic shocks. Let S be the finite set of realizations of exogenous asset return shocks and s_t denote the realization of a shock at date $t \geq 1$. The dated sequence of realizations of shocks facilitates the identification of optimal consumption and asset demands associated with each date and risky asset return realization. Thus, we introduce a history of realizations of shocks up to some date t

$$s^t = (s_1, s_2, \ldots, s_t) \in \underbrace{S \times ... \times S}_{t\ times} = S^t$$

which is also referred to as a date-event.[8] Since each chance node in a tree can be reached only through one historical path, s^t also uniquely defines a chance node and corresponding consumption and asset demands.[9] If s^τ precedes s^t in a tree, then we write $s^t \succ s^\tau$.

Let $c(s^t) \in C \subseteq \mathbb{R}_+$ denote consumption at node s^t and $\mathbf{c} = (c(s^1), \{c(s^2)\}, ..., \{c(s^T)\})$, where $s^t \in S^t$ and $t \in \{1, ..., T\}$, denote the T-period consumption vector. Individuals have preferences over the set of vectors $\mathbf{c}$ which are represented by the utility function $\mathcal{U}(\mathbf{c})$. As will be clear from the context, throughout this paper $\mathbf{c}$ will be used to denote both the dynamic consumption vectors and the consumption trees corresponding to the consumption vectors, where the set of asset payoff shocks S and branch probabilities are determined by the assumed asset payoff distribution.

We next briefly describe the DOCE utility axiomatized in Selden and Stux (1978). Assume a T period setting, where consumption in period $t = 1$ is certain and risky in periods $t = 2, ..., T$. In period t, the consumer's certainty time preferences over degenerate consumption streams $(c_t, ..., c_T)$ $(t \in \{1, ..., T\})$ are

[8]The shock s_1 is degenerate in the sense that it is the result of an outcome prior to period $t = 1$.

[9]To illustrate the application of the dated history of shocks, note that in the three period tree in Figure 1, the node associated with the planned consumption c_{31} is defined by the realization of the asset return shocks s_1, s_2 and s_3. This node corresponding to the upper branch of the tree and the associated consumption can only be reached by the history $s^3 = (s_1, s_2, s_3)$.

represented by the following additively separable utility

$$U_t(c_t, ..., c_T) = u(c_t) + \sum_{i=t+1}^{T} \beta^{i-t} u(c_i), \quad (1)$$

where $0 < \beta \leq 1$ is the standard discount function. The consumer's risk preferences in each period $t \in \{2, ..., T\}$ are identical and represented by the single period EU function

$$\sum_{s^t} \pi(s^t) V(c(s^t)), \quad (2)$$

where $\pi(s^t)$ is the probability of the date-event (node) s^t and V is the NM (von Neumann-Morgenstern) index. The stationary time preference u and NM index V will be assumed to satisfy $u' > 0, u'' < 0, V' > 0$ and $V'' < 0$ unless stated otherwise. In what follows, we use preferences over current and future consumption conditional on the current date-event node being s^τ.

The period t certainty equivalent evaluated at node s^τ is defined by

$$(\widehat{c}_t | s^\tau) = V^{-1} \left(\sum_{s^t \succ s^\tau} \pi(s^t | s^\tau) V(c(s^t)) \right), \quad (3)$$

where $\pi(s^t|s^\tau)$ is the probability of date-event s^t conditional on being at node s^τ. Thus, for a given s^τ, the DOCE utility for the consumption tree $\mathbf{c}$ is given by

$$\mathcal{U}(\mathbf{c}|s^\tau) = u(c(s^\tau)) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\widehat{c}_t | s^\tau).$$

Note that $\mathcal{U}(\cdot|s^\tau)$ is a function of $\mathbf{c}$ but only varies with $c(s^\tau)$ and $c(s^{\tau+i})$, $i = 1, ..., T-\tau$, where $s^{\tau+i} \succ s^\tau$. In period 1, the utility is given by

$$\mathcal{U}(\mathbf{c}) = u(c_1) + \sum_{t=2}^{T} \beta^{t-1} u(\widehat{c}_t | s^1). \quad (4)$$

For the DOCE preference model, (i) risk preferences are constant over time, (ii) there is a complete separation of time and risk preferences corresponding to U

and V [10] and (iii) the consumer is psychically indifferent to when risk is resolved (see the discussion of TRI below).

Kreps and Porteus (1978) derived the recursive representation

$$\mathcal{U}(\mathbf{c}|s^\tau) = U\left(c(s^\tau), \sum_{s^{\tau+1}\succ s^\tau} \pi(s^{\tau+1}|s^\tau)\mathcal{U}(\mathbf{c}|s^{\tau+1})\right),$$

where U is continuous and strictly increasing.[11] Note that if U is linear in the second argument, the KP representation specializes to the EU special case. The EZW representation is a special case of the KP utility,[12] where

$$U(c_t, x) = -\frac{\left(c_t^{-\delta_1} + \beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \text{ and } V_T(x) = -\frac{x^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1), \tag{5}$$

with V_T being induced from $\mathcal{U}$.[13] If $\delta_1 = \delta_2 = \delta$, the EZW representation specializes to the EU function

$$\mathcal{U}(\mathbf{c}|s^\tau) = -\frac{(c(s^\tau))^{-\delta}}{\delta} - E\left[\sum_{t=\tau+1}^{T} \beta^{t-\tau}\frac{(c_t(s^\tau))^{-\delta}}{\delta}\right].$$

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[10]As discussed in Selden (1978), time and risk preferences satisfy SEP in the sense that (i) the OCE utility is constructed from the independent building blocks (U, V) and (ii) if a given general continuous and monotone utility satisfies the OCE axioms, then it is always possible to derive the unique, up to appropriate transformations, separate U and V indices.

[11]Unlike the DOCE case, the KP preference building blocks are U and $\mathcal{U}$. An EU index V can be induced from the KP utility for the final time period T.

[12]For simplicity, we refer to (5) as EZW utility. However, as noted by Weil (1990, footnote 9), his form differs from the Epstein and Zin (1989) expression (5).

[13]It should be noted that both Epstein and Zin (1989) and Weil (1990) give recursive forms of their utility and do not formally identify the time and risk preference utilities defined over consumption as in (5). In fact, in their representations, risk preferences are defined over utility values rather than consumption values.

Both the KP and EZW recursive preference structures can accommodate a preference for early or late resolution of risk. However, as mentioned in the prior section, this temporal resolution preference cannot be varied independently from time and risk preferences.

In this paper, we assume DOCE preferences are based on the following time and risk preference building block utilities which results in the preferences being homothetic[14]

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 > -1, \delta_1, \delta_2 \neq 0) \tag{6}$$

$$u(c) = \ln(c) \qquad \text{and} \quad V(c) = \ln(c) \qquad (\delta_1, \delta_2 = 0). \tag{7}$$

For the DOCE and KP preferences based on (6)-(7), the EIS and Arrow-Pratt RRA (relative risk aversion) measures are given by, respectively,

$$EIS = \frac{1}{1+\delta_1} \quad \text{and} \quad -c_t \frac{V''(c_t)}{V'(c_t)} = 1 + \delta_2. \tag{8}$$

Before concluding this subsection, we discuss TC, TRI and SEP and then review what is known about their simultaneous satisfaction for the DOCE and KP preference models. Given the different axioms of these two models, it will prove more tractable in facilitating comparisons to state the following properties in terms of utility functions rather than the more foundational preferences.

The classic meaning of time consistency is illustrated in Figure 2. Consider the two three period, four branch consumption trees which are identical except for their respective second period lower branch subtrees. The subtrees differ in both their consumption values and probabilities as indicated by the unprimed and primed values. In Figure 2, time consistency requires that if the three period tree with the unprimed subtree is preferred to the three period tree with the primed subtree, then the unprimed period two continuation must be preferred to the

[14] To avoid corner solutions for the consumption-portfolio problems considered below, we do not include the $\delta_2 = -1$ case.

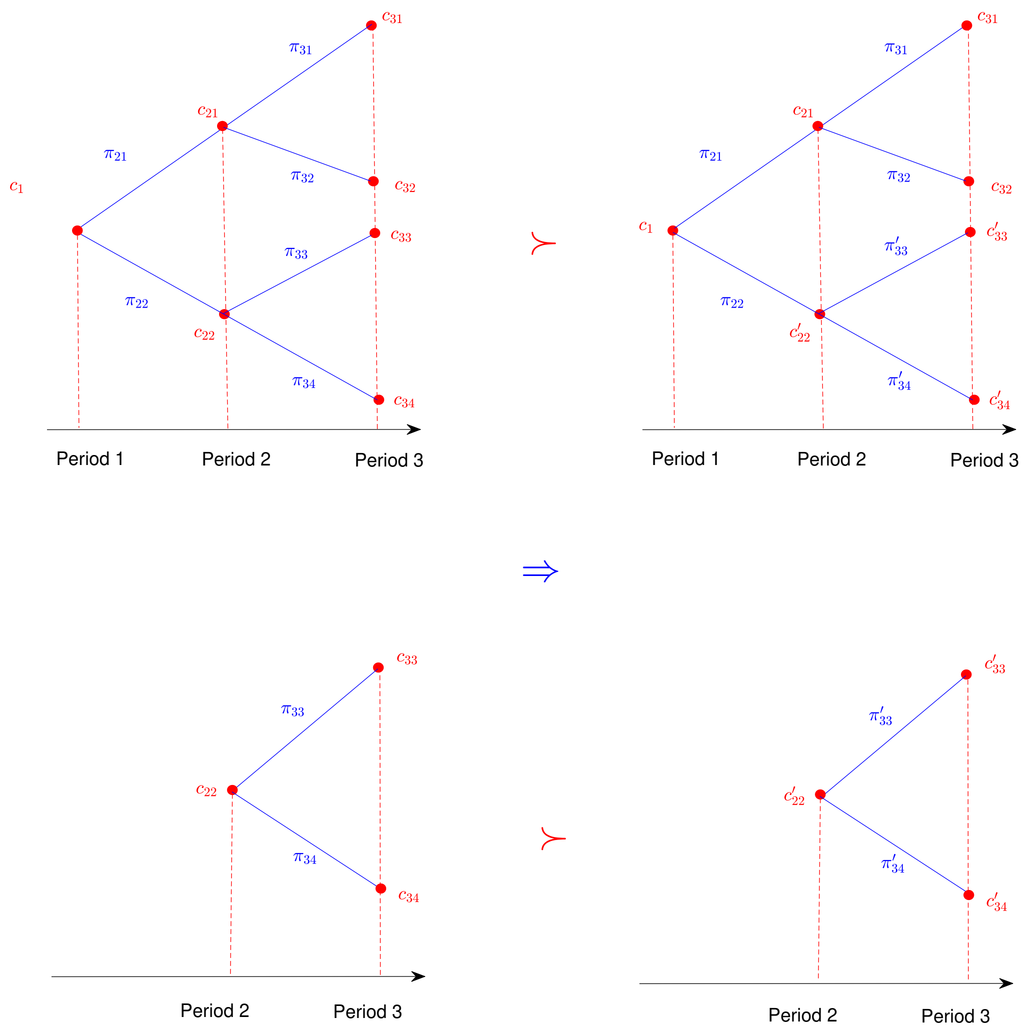


FIGURE 2. Illustration of Time Consistency

primed continuation. More formally, we employ the following version of the TC definition in Johnsen and Donaldson (1985).[15]

DEFINITION 1. The consumer's preferences satisfy TC if and only if at any time t with some payoff history s^t, if the two trees $\mathbf{c}$ and $\mathbf{c}'$ only differ in a subtree starting from a given note $s^{t+1} \succ s^t$, then

$$\mathcal{U}\left(\mathbf{c}|s^t\right) \geq \mathcal{U}\left(\mathbf{c}'|s^t\right) \Rightarrow \mathcal{U}\left(\mathbf{c}|s^{t+1}\right) \geq \mathcal{U}\left(\mathbf{c}'|s^{t+1}\right).$$

[15] See a similar definition in Figure 2 in Epstein and Zin (1989, p. 945).

The preference for when risk is resolved over time is a key feature of KP preferences and the special case of temporal resolution indifference is a central element in the axiomatization of DOCE preferences in Selden and Stux (1978). In this paper, our primary interest in TRI relates to its ruling out the implausibly large timing premium in some asset pricing and macro models based on the Epstein and Zin (1989) special case of KP preferences. Epstein, Farhi and Strzalecki (2014) consider the case of multiple consumption processes. Let $(\mathbf{c}, \boldsymbol{\pi})$ denote a process where $\mathbf{c}$ is the vector of consumption over time and different states and $\boldsymbol{\pi}$ denotes the associated state probabilities. To define the timing premium for the consumption processes $(\mathbf{c}, \boldsymbol{\pi})$, Epstein, Farhi and Strzalecki (2013) consider the special process $(\mathbf{c}', \boldsymbol{\pi}')$ which is derived from $(\mathbf{c}, \boldsymbol{\pi})$ and has the same period one probability distribution over subsequent consumption but differs in having all of the risk resolved at the beginning of period two. One can view the derived process $(\mathbf{c}', \boldsymbol{\pi}')$ as corresponding to an alternative tree which has S^{T-1} branches in period 1 with no uncertainty at periods $t = 2, \ldots, T$. Each branch in the first period can then be associated with a terminal node of the original tree. Define consumption of the tree corresponding to $(\mathbf{c}', \boldsymbol{\pi}')$ by $c_t'(s^T) = c_t(s^T)$ and define probabilities as $\pi'(s^T) = \pi(s^T)$. (See Figure 3 where the consumption trees in (a) and (b) correspond, respectively, to the three period consumption processes $(\mathbf{c}, \boldsymbol{\pi})$ and $(\mathbf{c}', \boldsymbol{\pi}')$.)

Then we have the following definition.

Definition 2. The consumer's preferences satisfy TRI if and only if any consumption process $(\mathbf{c}, \boldsymbol{\pi})$ is indifferent to the corresponding early resolution tree $(\mathbf{c}', \boldsymbol{\pi}')$.

Note that this definition of TRI differs from (i) the TRI axiom in Strzalecki (2013) due in part to his consideration of the case of ambiguity preferences and (ii) the TRI axiom in Selden and Stux (1978) since they consider not just the early resolution tree but also other trees with partial resolution. In our definition, risk is resolved after period 1 while in the other two papers, it may be resolved at a later date. This difference turns out to play no role in our analysis.

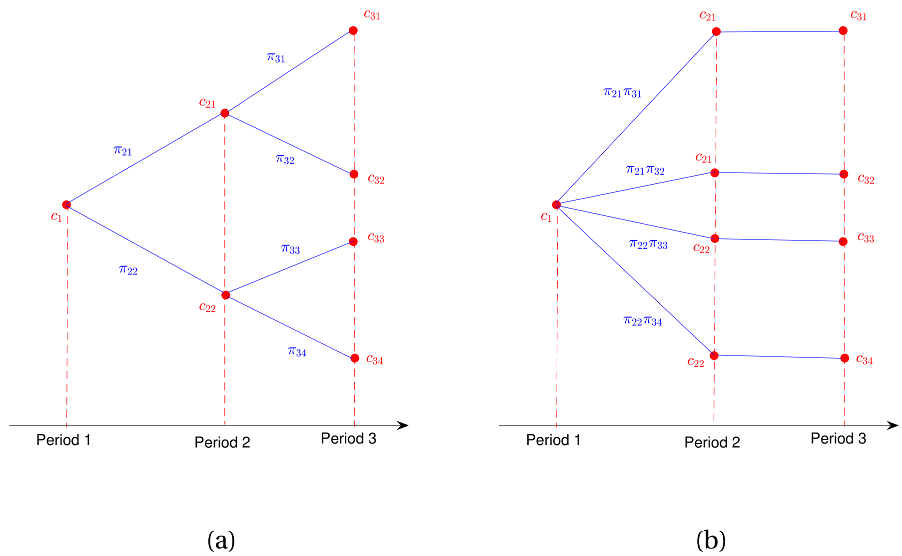


FIGURE 3. Illustration of TRI

REMARK 1. Following Epstein, Farhi and Strzalecki (2014, p. 2684), given a utility function U a timing premium ϑ is defined as

$$\vartheta = 1 - \frac{U\left(\mathbf{c}, \boldsymbol{\pi}\right)}{U\left(\mathbf{c}', \boldsymbol{\pi}'\right)}.$$

Therefore, preferences satisfy TRI if and only if for any consumption processes $(\mathbf{c}, \boldsymbol{\pi})$, the timing premium is zero.

Finally the dynamic utility defined over consumption trees $\mathcal{U}(\mathbf{c})$ and its underlying preferences will be said to exhibit SEP if it is based on the two independent building blocks (U, V). This definition is clearly satisfied by DOCE preferences. However, in the case of KP preferences, assuming a psychic preference for early or late resolution imposes a restriction on U and V and hence violates the independence of the time and risk preference indices.[16] To see this most clearly, consider the case where the building block utilities correspond to CES time and CRRA risk preference utilities (6)-(7). In this case, as observed by Epstein and

[16] The EU representation is a special case of both the DOCE and KP preference models where u and V are positive affine transforms of each other and hence SEP is violated.

Zin (1989, p.952), the consumer has a preference for the early (late) resolution tree depending on whether her risk preference parameter $\delta_2 > (<)$ her time preference parameter δ_1. Thus if a KP consumer has a preference for the early resolution tree, her U and V building blocks cannot be prescribed independently as they must satisfy $\delta_2 > \delta_1$.[17]

In terms of simultaneously satisfying TRI and SEP, Kreps and Porteus (1978, Corollary 3) prove that TRI implies that their recursive utility specializes to a dynamic EU representation. Since the EU function does not satisfy SEP, this implies the impossibility of satisfying these two properties at the same time.[18] In this paper, we argue that although it is impossible to satisfy TRI, SEP and TC simultaneously at the level of preferences for general choice spaces, the three properties can be satisfied in the context of the joint consumption-portfolio optimization problem introduced in the next subsection when appropriate restrictions are imposed on asset returns and the time and risk preference building block utilities.

2.2 *Optimization Problem*

In this subsection, we formally define the consumption-portfolio problem.

At the beginning of each period $t = 1, \ldots, T-1$ there are J one period assets available for trade with returns $\mathbf{R}\left(s^{t+1}\right) = \left(R_j\left(s^{t+1}\right)\right)_{j=1}^{J} \geq 0$ being realized at node s^{t+1}. We assume that asset returns preclude arbitrage and $\mathbf{R}\left(s^{t+1}\right)$ has full rank J. It is a basic result in finance[19] (see, e.g., Duffie (2001, p. 3)) that this is equivalent to the existence of $\rho(s^t) > 0$ for all s^t such that

$$\sum_{s^{t+1} \succ s^t} \rho(s^{t+1}) R_j(s^{t+1}) = \rho(s^t) \ \ \forall s^t, j. \tag{9}$$

[17] This point is noted in Strzalecki (2013, pp. 1056-1057) and Epstein, Farhi and Strzalecki (2014, p. 2688).

[18] Strzalecki (2013) also considers axioms similar to TRI, SEP and TC. Starting from discounted uncertainty averse preferences, his Theorem 1 proves that the corresponding axioms are simultaneously satisfied if and only if one has a dynamic maxmin EU model.

[19] In the special case of complete asset markets, where the number of assets is the same as the number of states, or more formally, where at each s^t, $t < T$, the matrix $(R(s^{t+1}))_{\{s^{t+1} \succ s^t\}}$ has rank S the $\rho(s^{t+1})$ in eqn. (9) can be interpreted as the contingent claim price for $c(s^{t+1})$.

An individual is assumed to choose consumption and assets in periods $t = 1, \ldots, T-1$ so as to maximize utility. We assume throughout that the individual has rational expectations in that she knows future asset returns contingent on the nodes.

In period $t \in \{1, ..., T-1\}$, at the node s^t, denote the demand for asset $j \in \{1, ..., J\}$ by $n_j\left(s^t\right)$ and the vector of asset holdings by $\mathbf{n} = (\mathbf{n}\left(s^1\right), ..., \left\{\mathbf{n}\left(s^t\right)\right\}, ..., \left\{\mathbf{n}\left(s^{T-1}\right)\right\})$, where $\mathbf{n}\left(s^t\right) = (n_1\left(s^t\right), ..., n_j\left(s^t\right))$. Let I denote initial income.

The period 1 consumption-portfolio problem is defined as follows

$$\max_{\mathbf{c},\mathbf{n}} \mathcal{U}\left(\mathbf{c}\right) \quad S.T. \tag{10}$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = 1, \tag{11}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2 < t < T, \tag{12}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T. \tag{13}$$

In the certainty case, Blackorby, et al. (1973) prove that demands are time consistent if and only if each period $t+1$ utility can be embedded into the period t utility for all $t \in \{1, ..., T-1\}$ utilities. Johnsen and Donaldson (1985) extend this notion to the risky case, where time consistency holds if and only if the future utility function in each state can be embedded into the utility function of prior periods. If time consistency fails to hold, then one needs to consider resolute, naive and sophisticated choice.[20]

For resolute choice, the period 1 consumption-portfolio problem is defined as follows

$$\max_{\mathbf{c},\mathbf{n}} \mathcal{U}\left(\mathbf{c}\right) \quad S.T. \tag{14}$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = 1, \tag{15}$$

[20] This is standard in the certain intertemporal setting (e.g., Selden and Wei (2016, p. 1916)).

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2 < t < T, \tag{16}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T. \tag{17}$$

In this case, the consumer maximizes her period one utility $\mathcal{U}(\mathbf{c})$ and $\left(\mathbf{c}^R, \mathbf{n}^R\right)$ denotes the optimal resolute demands.

For naive choice, we have $c^N(s^1) = c^R(s^1)$ and $\mathbf{n}^N(s^1) = \mathbf{n}^R(s^1)$. Then in each period τ $(2 \leq \tau < T)$, $c^N(s^\tau)$ and $\mathbf{n}^N(s^\tau)$ are derived based on the period τ utility function for the given $\mathbf{n}^N(s^{\tau-1})$, i.e.,

$$\max_{\mathbf{c},\mathbf{n}} \mathcal{U}(\mathbf{c}|s^\tau) \quad S.T. \tag{18}$$

$$c(s^\tau) = \mathbf{n}(s^{\tau-1}) \cdot \mathbf{R}(s^\tau) - \sum_j n_j(s^\tau). \tag{19}$$

As implied by the maximization of $U(\mathbf{c}|s^\tau)$, in each period after the first the naive consumer reoptimizes.

For sophisticated choice, define for each s^T

$$c^0(s^T, \overline{\mathbf{n}}) = \overline{\mathbf{n}} \cdot \mathbf{R}(s^T),$$

and for each s^τ, $\tau = 2, \ldots, T-1$ recursively,

$$(c^0, \mathbf{n}^0)(s^\tau, \overline{\mathbf{n}}) = \arg\max_{(c,\mathbf{n})} \mathcal{U}(\mathbf{c}|s^\tau) \quad S.T. \tag{20}$$

$$c(s^\tau) = \overline{\mathbf{n}} \cdot \mathbf{R}(s^t) - \sum_j n_j, \tag{21}$$

$$c(s^{\tau+i}) = c^i(s^{\tau+i}, \mathbf{n}), \ i = 1, ..., T-\tau, \tag{22}$$

where, also recursively, for all s^t

$$c^i(s^t, \overline{\mathbf{n}}) = c^{i-1}(s^t, \mathbf{n}^{i-1}(s^{t-1}, \overline{\mathbf{n}})), \tag{23}$$

and

$$\mathbf{n}^i(s^t, \overline{\mathbf{n}}) = \mathbf{n}^{i-1}(s^t, \mathbf{n}^{i-1}(s^{t-1}, \overline{\mathbf{n}})). \tag{24}$$

Optimal sophisticated choice can then be computed by iterating forward,

$$(c(s^t), \mathbf{n}(s^t)) = (c^0, \mathbf{n}^0(s^t, \mathbf{n}(s^{t-1}))), \tag{25}$$

3. Time Consistent DOCE Demand

In this section, we derive necessary and sufficient conditions such that CES-CRRA DOCE preferences are time consistent. We also show that under these conditions, the DOCE utility can be used to extend the classic certainty Fisherian consumption-saving analysis to a T-period dynamic consumption-portfolio setting.

3.1 *Time Consistent Preferences over Restricted Domains*

In this first subsection, we provide necessary and sufficient conditions on consumption sets such that DOCE preferences are time consistent. In the next subsection, we state necessary and sufficient conditions on asset returns ensuring that choices always lie in this set.

It is well-known that in general, DOCE preferences violate time consistency as defined in Definition 1 above. We argue in this subsection that if one restricts the domain of preferences (i.e., assumes that possible choices have to lie in a subset of all possible consumption choices on the event-tree), time consistency can be restored.

Before stating the general result, Proposition 1, characterizing restrictions on the domain of preferences that ensure that DOCE preferences are time consistent, we first provide a transparent example for why this can happen. For simplicity, assume the simple three period consumption tree in Figure 1. Given the fixed tree structure in Figure 1 and set of probabilities, a given consumption tree can be fully characterized by the consumption vector

$$\mathbf{c} = (c_1, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, c_{34}) \in \mathbb{R}^7_+.$$

The vectors $(c_{21}, c_{31}, c_{32}) \in \mathbb{R}^3_+$ and $(c_{22}, c_{33}, c_{34}) \in \mathbb{R}^3_+$, respectively, characterize in a natural way the upper and lower subtrees. Let $\mathcal{U}(c_{21}, c_{31}, c_{32}|c_1)$ and

$\mathcal{U}(c_{22}, c_{33}, c_{34}|c_1)$ represent, respectively, the DOCE preferences over the consumption on the subtrees corresponding to (c_{21}, c_{31}, c_{32}) and (c_{22}, c_{33}, c_{34}). In the current setting, the TC Definition 1 then simplifies to the following. For all $\mathbf{c}$, $\mathbf{c}'$ with $c_1 = c_1'$,

$$\mathcal{U}(c_{21}, c_{31}, c_{32}|c_1) \geq \mathcal{U}\left(c_{21}', c_{31}', c_{32}'|c_1\right) \text{ and } \mathcal{U}(c_{22}, c_{33}, c_{34}|c_1) \geq \mathcal{U}\left(c_{22}', c_{33}', c_{34}'|c_1\right)$$
$$\Longrightarrow \mathcal{U}(\mathbf{c}) \geq \mathcal{U}\left(\mathbf{c}'\right). \tag{26}$$

While DOCE preferences do not satisfy TC over $\mathbb{R}_+^7$, it is easy to see that for any $\bar{\mathbf{c}} \in \mathbb{R}_+^7$ they are time consistent over $\{\mathbf{c} \in \mathbb{R}_+^7 : \mathbf{c} = \alpha\bar{\mathbf{c}}, \alpha > 0\}$ whenever they are strongly monotone. This trivial example illustrates that time consistency is a joint property of preferences and the domain over which they are defined.

Recognizing that the assumed CES-CRRA DOCE preferences are homothetic, consider the following set as the domain of preferences.

$$\mathcal{I} = \left\{ \begin{array}{c} \mathbf{c} \in \mathbb{R}_+^7 : \mathbf{c} = (c_1, c_{21}, c_{22}, \alpha_1 c_{21}, \alpha_2 c_{21}, \alpha_3 c_{22}, \alpha_4 c_{22}), \\ (\alpha_1, \ldots, \alpha_4) \in \mathbb{R}_+^4, \pi_{31}\alpha_1^{-\delta_2} + \pi_{32}\alpha_2^{-\delta_2} = \pi_{33}\alpha_3^{-\delta_2} + \pi_{34}\alpha_4^{-\delta_2} \end{array} \right\}. \tag{27}$$

We next argue that the DOCE preferences are TC over the domain $\mathcal{I}$. Define

$$K = \pi_{33}\alpha_3^{-\delta_2} + \pi_{34}\alpha_4^{-\delta_2}$$

and rewrite the period 1 utility function as follows

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= u(c_1) + \beta u \circ V^{-1}\left(\sum_{i=1}^{2} \pi_{2i} V(c_{2i})\right) + \beta^2 u \circ V^{-1}\left(\sum_{i=1}^{2} \pi_{2i} \sum_j \pi_{3j} V(\alpha_j c_{2i})\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_i \pi_{2i} V(c_{2i})\right) + \beta^2 \left(u \circ V^{-1}\left(\sum_i \pi_{2i} V(c_{2i})\right) u \circ V^{-1}(K)\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_i \pi_{2i} V(c_{2i})\right)\left(1 + \beta u \circ V^{-1}(K)\right)
\end{aligned}$$

and depending on whether the upper or lower state is realized

$$\mathcal{U}\left(c_{21},c_{22},c_{31},c_{32},c_{33},c_{34}|c_1\right)=u(c_{2i})+\beta u\circ V^{-1}\left(\sum_j \pi_{3j}V\left(\alpha_j c_{2i}\right)\right)$$
$$=u(c_{2i})\left(1+\beta u\circ V^{-1}\left(K\right)\right)\ (i=1,2).$$

It is now easy to see that homothetic DOCE preferences are time consistent over the domain $\mathcal{I}$ since the eqn. (26) condition for TC holds. The following proposition generalizes this result to arbitrary date event consumption trees.

PROPOSITION 1. *Assume CES-CRRA DOCE preferences which are defined over consumption on a date-event consumption tree with M nodes. Then the preferences are time consistent if and only if consumption is restricted to the set*

$$\mathcal{I}=\left\{\begin{array}{c}\mathbf{c}\in\mathbb{R}^M:\exists V_t, t=2,...,T-1,\ \textit{such that}\\ \forall s^t, t<T, \frac{-1}{\delta_2}\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)\left(\frac{c(s^{t+1})}{c(s^t)}\right)^{-\delta_2}=V_t\end{array}\right\}. \tag{28}$$

To see that eqn. (28) converges to eqn. (27) corresponding to the tree in Figure 1, note that in eqn. (28), for the consumption subtrees corresponding (c_{21},c_{31},c_{32}) and (c_{22},c_{33},c_{34}) we have, respectively,

$$\frac{-1}{\delta_2}\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)\left(\frac{c(s^{t+1})}{c(s^t)}\right)^{-\delta_2}=-\frac{1}{\delta_2}\left(\pi_{31}\left(\frac{c_{31}}{c_{21}}\right)^{-\delta_2}+\pi_{32}\left(\frac{c_{32}}{c_{21}}\right)^{-\delta_2}\right)=V_2$$

and

$$\frac{-1}{\delta_2}\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)\left(\frac{c(s^{t+1})}{c(s^t)}\right)^{-\delta_2}=-\frac{1}{\delta_2}\left(\pi_{33}\left(\frac{c_{33}}{c_{22}}\right)^{-\delta_2}+\pi_{34}\left(\frac{c_{34}}{c_{22}}\right)^{-\delta_2}\right)=V_2,$$

implying that

$$-\frac{1}{\delta_2}\left(\pi_{31}\left(\frac{c_{31}}{c_{21}}\right)^{-\delta_2}+\pi_{32}\left(\frac{c_{32}}{c_{21}}\right)^{-\delta_2}\right)=-\frac{1}{\delta_2}\left(\pi_{33}\left(\frac{c_{33}}{c_{22}}\right)^{-\delta_2}+\pi_{34}\left(\frac{c_{34}}{c_{22}}\right)^{-\delta_2}\right). \tag{29}$$

Then in eqn. (27), since

$$\pi_{31}\left(\frac{c_{31}}{c_{21}}\right)^{-\delta_2}+\pi_{32}\left(\frac{c_{32}}{c_{21}}\right)^{-\delta_2}=\pi_{31}\alpha_1^{-\delta_2}+\pi_{32}\alpha_2^{-\delta_2}$$

and

$$\pi_{33}\left(\frac{c_{33}}{c_{22}}\right)^{-\delta_2}+\pi_{34}\left(\frac{c_{34}}{c_{22}}\right)^{-\delta_2}=\pi_{33}\alpha_3^{-\delta_2}+\pi_{34}\alpha_4^{-\delta_2},$$

eqn. (29) implies that

$$\pi_{31}\alpha_1^{-\delta_2}+\pi_{32}\alpha_2^{-\delta_2}=\pi_{33}\alpha_3^{-\delta_2}+\pi_{34}\alpha_4^{-\delta_2},$$

which is the requirement in eqn. (27).

3.2 *Main Result*

To apply Proposition 1 to the consumption-portfolio optimization problem (10) - (13), we next provide simple conditions on asset returns that ensure that an individual's optimal consumption $\mathbf{c}$ lies in the set $\mathcal{I}$. In order to do so, it is useful to define for each s^t, $\widetilde{\mathbf{n}}(s^t)\in\mathbb{R}^J$ to be the unique solution to the J equations

$$\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)R(s^{t+1})V'\left(R(s^{t+1})\cdot\widetilde{\mathbf{n}}(s^t)\right)=1, \tag{30}$$

where the homothetic risk preference NM index takes the CRRA form in (6)-(7).

The following identical certainty equivalent return assumption then plays a key role in the time consistency of DOCE choice.

Assumption ICER Assume that for all s^t, $t<T$,

$$\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)\left(R(s^{t+1})\cdot\widetilde{\mathbf{n}}(s^t)\right)^{-\delta_2}=\left(\widehat{R}_{pt+1}\sum_j\widetilde{\mathbf{n}}_j(s^t)\right)^{-\delta_2}, \tag{31}$$

where $\widehat{R}_{pt+1}$ is non-stochastic.[21]

This assumption begins with the certainty equivalent return $\widehat{R}_{pt+1}$ of the optimal asset portfolio for the one period subtree starting from chance node s^t and requires that for all chance nodes s^t in period t, the certainty equivalent return

[21] $\widehat{R}_{pt+1}$ is the certainty equivalent return of the optimal asset portfolio. Since the preferences are homothetic, it is always possible to obtain a non-stochastic certainty equivalent return. Also refer to Selden and Wei (2024) for the discussion of $\widehat{R}_p$ in a two period case.

of the optimal portfolio is the same. However, ICER allows for different return distributions in different time periods.

REMARK 2. It should be emphasized that the ICER property does not depend on the asset choices, but more fundamentally it depends on the asset return distribution characteristics, probabilities and preference parameters such as the value of δ_2.

When markets are incomplete this assumption can be difficult to verify. However, a simple sufficient condition for Assumption ICER to hold is that the asset return distributions are identical across all nodes in a given period.

When markets are complete, Assumption ICER can be directly translated into an assumption on asset returns. Based on the assumed CRRA representation of risk preferences, corresponding to different values of δ_2, different sets of asset return distributions will satisfy ICER. Each branch in period t has an asset return distribution from the same set parameterized by $\widehat{R}_{pt}$.

We have the following result.

THEOREM 1. *Suppose the consumer solves the consumption-portfolio problem (10) - (13). Then the following hold.*

(i) *If the consumer's DOCE utility takes the form corresponding to the CES-CRRA utilities*

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 > -1, \delta_1, \delta_2 \neq 0),$$

her demands will be time consistent if and only if Assumption ICER holds.

(ii) *If Assumption ICER holds, then the consumer's demands will be time consistent if and only if her preferences are represented by the DOCE utility corresponding to*

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 > -1, \delta_1, \delta_2 \neq 0).$$

At first glance, the theorem seems very surprising: Why is the portfolio certainty equivalent return being non-stochastic sufficient for DOCE preferences to be time consistent? While the detailed proof of Theorem 1 gives a formal answer to this puzzle, the key insight is that Assumption ICER simplifies the first order conditions and implies the conditions in Proposition 1. The intuition for why in Theorem 1(i) Assumption ICER is sufficient for TC to hold follows directly from its interaction with the implications of CRRA risk preferences and CES time preferences. Without loss of generality, consider the three period tree in Figure 1. Let I_{21} and I_{22}, respectively, denote the period 2 income at the nodes corresponding to the upper and lower branches based on the optimal period 1 demands (c_1, n_1, n_{f1}). At the period 2 nodes, CRRA risk preferences imply one fund separation and the certainty equivalent period 3 consumption constraint can be written as

$$\widehat{c}_{3i} = (I_{2i} - c_{2i})\widehat{R}_{p2} \ \ (i = 1, 2)$$

where i denotes the period 2 node and Assumption ICER implies that the portfolio certainty equivalent return $\widehat{R}_{p2}$ is the same at both two nodes. Given these two linear constraints, the consumer confronts a standard consumption-saving problem associated with selecting the optimal c_{2i}. The assumption of CES time preference implies that the fractions of consumption c_{2i}/I_{2i} and saving $(I_{2i} - c_{2i})/I_{2i}$ are independent of I_{2i}. Thus, since Assumption ICER implies that $\widehat{R}_{p2}$ will be the same at both nodes and c_{2i}/I_{2i} and $(I_{2i} - c_{2i})/I_{2i}$ will also be the same at both nodes, the consumer will have no reason change her period 2 consumption portfolio solution conditional on which node is realized and hence her demands will be time consistent.[22]

In the application of Theorem 1, Assumption ICER guarantees that optimal demands at time 1 will lie in set $\mathcal{I}$ and one can restrict the consumer's choice to just $\mathcal{I}$ rather than to the full budget set. The reader may wonder whether this is also true with the passage of time, say to period 2. Suppose we define the "continuation" set $\mathcal{I}_2$ where c_1 has been realized and part of initial set $\mathcal{I}$ corresponding to nodes associated with states that can no longer be realized have been deleted.

[22]See the related discussion in the next subsection concerning the decomposition of the consumption-portfolio problem into an equivalent conditional portfolio problem and consumption-saving problem.

Now ICER guarantees that no new information has been realized with the passage from time period 1 to time period 2 and so the same argument that is used for Theorem 1 can be applied to ensure that the period 1 optimal continuation consumption plan continues to be optimal in the continuation consumption set $\mathcal{I}_2$. More formally this can be seen from eqn. (34) in the proof of Theorem 1. Since we have time consistency, this is equivalent to a consumer following naïve choice not revising her resolute choice and corresponding plan derived in period 1.

REMARK 3. We give conditions in Supplemental Appendix A such that a consumer with HARA (hyperbolic absolute risk aversion) DOCE preferences exhibits TC.

3.3 *A Risky Extension of the Classic Certainty Fisherian Consumption-Saving Analysis*

In this subsection, we explain how the assumptions of CES-CRRA DOCE utility and ICER asset returns facilitate a natural risky portfolio extension of the classic Fisherian certainty two period diagrammatic analysis frequently used in macroeconomic textbooks to introduce consumption and saving (e.g., Romer (2006) and Mankiw (2010)).[23] We then consider the extension to the T-period case.

For the consumption-portfolio problem at any date $t = 1, ..., T-1$ node, certainty equivalent consumption at date $t+1$, $\widehat{c}_{t+1}$, can be expressed as the product of date t saving times a certainty equivalent portfolio return $\widehat{R}_{pt}$. Because this return is constant for all levels of saving, the consumer's two period consumption-saving problem is exactly analogous to the Fisherian certainty case where the certainty equivalent return plays the same role as the risk free rate. See Figure 4, where I_t denotes period t initial income, $I_t - c_t$ denotes period t saving and $U = const$ specifies a certainty time preference indifference curve. The same restrictions on the EIS that ensure that saving increases with a decrease in the risk free rate in a certainty setting also guarantee that saving increases with a

[23] Although the Fisherian diagrammatic analysis is typically dropped once risk is introduced, we show how it can be extended to the case of risky saving by exploiting the DOCE preference model.

(mean preserving) increase in risk and a reduction in $\widehat{R}_{pt}$. This pedagogical tool highlights the complete separation of time and risk as reflected, respectively, in the time preference indifference curves and the certainty equivalent budget constraint.[24] Much of the two period Fisherian analysis extends to the T-period setting assumed in Theorem 1, where the risk free rate in each period R_{ft} is replaced by the certainty equivalent return $\widehat{R}_{pt}$.

4. KP Rationalization

Suppose that KP preferences are constructed from the same CES-CRRA time and risk preference building block utilities as in the time consistent DOCE case and one assumes that asset returns satisfy ICER. Quite surprisingly, we next show that the two preference relations which are not ordinally equivalent over the full choice space, nevertheless result in the same demands.

Proposition 2. *Suppose Assumption ICER holds and the consumer has DOCE utility corresponding to the CES-CRRA form (6)-(7) and solves the consumption-portfolio problem (10) - (13). Then the optimal demands can also be rationalized by KP preferences, where*

$$U\left(c_t, x\right) = -\frac{\left(c_t^{-\delta_1} + \beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \text{ and } V_T\left(x\right) = -\frac{x^{-\delta_2}}{\delta_2}.$$

The proof of Proposition 2 shows that if restricted to the consumption set $\mathcal{I}$ as defined by (28), KP and DOCE preferences coincide. We then show that under Assumption ICER, the optimal choice for KP utility lies in $\mathcal{I}$.

To see the intuition for why the two utilities are identical for consumption in $\mathcal{I}$, consider the three period case in Figure 1. Note that since the period 2 DOCE optimization problem following backward induction is the same as the recursive KP optimization assuming ICER holds, the period 2 portfolio certainty equivalent

[24] It is worth emphasizing, as noted in Kimball and Weil (2009), that the DOCE preference model can sometimes provide more intuitive comparative statics results than the KP formulation given that the former is defined over $(c_1, \widehat{c}_2)$ rather than (c_1, EV) in the latter.

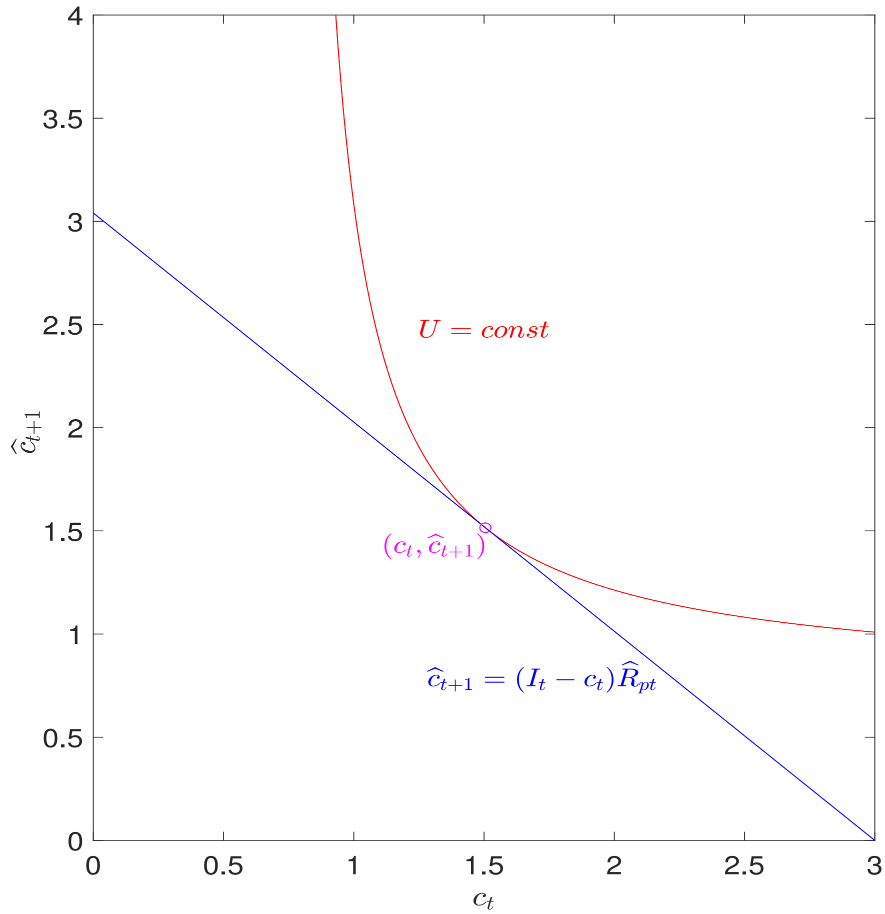


FIGURE 4. Fisherian Risky Consumption-Saving Problem

return $\widehat{R}_{p3}$ is the same across branches and we have

$$\widehat{c}_{31} = \alpha c_{21} \quad \text{and} \quad \widehat{c}_{32} = \alpha c_{22},$$

where $\alpha = \beta^{\frac{1}{1+\delta_1}} \widehat{R}_{p3}^{\frac{1}{1+\delta_1}}$.[25] Then for the KP case

$$\beta\left(\pi_1\left(c_{21}^{-\delta_1}+\beta\widehat{c}_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}+\pi_2\left(c_{22}^{-\delta_1}+\beta\widehat{c}_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}$$

$$=\beta\left(\pi_1\left(1+\beta\alpha^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}c_{21}^{-\delta_2}+\pi_2\left(1+\beta\alpha^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}$$

$$=\beta\left(1+\beta\alpha^{-\delta_1}\right)\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}$$

and for the DOCE case

$$\beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\left(\pi_1\widehat{c}_{31}^{-\delta_2}+\pi_2\widehat{c}_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}$$

$$=\beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\left(\pi_1\alpha^{-\delta_2}c_{21}^{-\delta_2}+\pi_2\alpha^{-\delta_2}c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}$$

$$=\beta\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}+\beta^2\alpha^{-\delta_1}\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}$$

$$=\beta\left(1+\beta\alpha^{-\delta_1}\right)\left(\pi_1 c_{21}^{-\delta_2}+\pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}},$$

which are the same.

The intuition for Proposition 2 is that when Assumption ICER holds, effectively no new information is received with the passage of time. Hence over the domain $\mathcal{I}$, DOCE like KP preferences satisfies time consistency. Also, the preference for early or late resolution exhibited by KP preferences cannot be distinguished from TRI for DOCE preferences over $\mathcal{I}$, since Assumption ICER rules out the canonical early resolution consumption tree (as for example in Figure 3(b)). Temporal resolution preferences require one to consider consumption trees outside the set $\mathcal{I}$.[26]

[25]Given that the conditions in Theorem 1 hold, DOCE preferences satisfy TC and one obtains the same solution using backward induction and the standard period 1 optimization.

[26]To see that an early resolution consumption tree such as Figure 3(b) cannot be in the set $\mathcal{I}$, consider the following example. Assume three time periods and DOCE preferences corresponding to

Does the fact that over the set $\mathcal{I}$ for the tree structure in Figure 1, KP utility becomes identical to DOCE utility which satisfies TRI obviate the concern raised in Section 1 associated with KP utility exhibiting a psychic preference for early or late resolution? Epstein, Farhi and Strzalecki (2014) raised the possibility that a KP consumer might be willing to pay an **implausibly** large timing premium to have periods $2, ..., T$ risks resolved in period 2. It should be emphasized that in this section as well as in the prior section, we have focused on demand behavior of the CES-CRRA DOCE and KP consumers where asset returns satisfy ICER. In contrast, the Epstein, Farhi and Strzalecki (2014) paper assumes an equilibrium asset pricing setting where assumptions are made on the exogenous consumption growth process rather than on asset returns. However, when they assume the consumption growth process is i.i.d., this implies that at the demand level, asset returns also satisfy i.i.d. (a special case of ICER). Under this condition, we have shown that the EZ (special case of KP preferences) and DOCE preferences are ordinally equivalent for the tree structure in Figure 1. It follows immediately that under the assumption of i.i.d. consumption growth, the equilibrium will be identical for economies based on an EZ versus DOCE representative consumer. However in this specific i.i.d. case, where the $EIS = 1.5$ and $RRA = 10$, Epstein, Farhi and Strzalecki (2014, p. 2690) computed a timing premium of approximately 9.5% for all of the risk to be resolved in the second period, whereas for the DOCE equilibrium there will be zero timing premium due to the assumption that TRI holds. For, the case of i.i.d. consumption growth, a timing premium

CES time and CRRA risk preferences. In period 1 there is only a risk free asset. In period 2, the consumer can invest in a risky and a risk free asset. Given this setup, the optimal consumption tree takes the form of a three period version of the late resolution tree in Figure 3(a). Since c_2 is certain, for the corresponding early resolution tree one must have $c_{21} = c_{22} = c_2$. Notice that in the early resolution tree, all risk is resolved at period two and hence in each sub-branch i, there exists only a risk free asset with the payoff R_{f3i}. Assumption ICER implies that $R_{f31} = R_{f32}$ and hence no matter how much is saved in period 1, period 2 income will be the same on the upper and lower branches. Since preferences are also the same on the upper and lower branches, optimal c_2 and c_3 will also be the same on the two branches. Thus the restricted domain $\mathcal{I}$ will necessarily exclude early resolution consumption trees with different c_3-values.

of 9.5% of all future consumption is certainly material, but far smaller than in the case of autocorrelated consumption growth.

Moreover, even if the timing premium becomes small for KP preferences, temporal resolution preferences still interact with SEP. Epstein and Zin (1989) note that for KP preferences a preference for early or late resolution of risk can confound the ability to fully satisfy SEP. Does the fact that KP and DOCE optimal demands converge when ICER or i.i.d. asset returns holds imply that the confounding can be avoided? Epstein and Zin (1989, p. 952) observe (restated in our notation) that δ_2 is interpreted as a risk preference parameter. Fixing the time preference parameter δ_1, an increase in δ_2 not only increases risk aversion but also can affect the KP temporal resolution preference. Epstein and Zin (1989, p. 952) conclude "One is left wondering how to interpret the comparative statics effects of a change in [δ_2]....[The EIS and RRA measures and attitudes toward timing] seem intertwined". The fact that a change in risk aversion has identically the same effects on DOCE and KP asset demands and saving when ICER holds, does not negate the Epstein and Zin point that a change in δ_2 could be the consequence of an increased preference for early resolution and not an increase in risk aversion. In contrast, because DOCE preferences satisfy TRI, the change in δ_2 is separate from the EIS and is independent of when risk is resolved.[27]

5. Relaxation of ICER

In this section, we relax the assumption that asset returns satisfy ICER (or i.i.d.). As a result, the DOCE consumer fails to satisfy time consistency and one needs to consider the resolute (naive) and sophisticated solution techniques to the intertemporal optimization problem defined toward the end of Subsection 2.2. When considering a perturbation to asset returns which causes ICER to fail, the

[27] In Supplemental Appendix B, we provide a specific example, in the context of the classic consumption-portfolio problem, where there is a confounding of time and risk preferences when EZW preferences are assumed but not when DOCE preferences are assumed.

optimal resolute and sophisticated consumption and asset demands will in general diverge. To quantify the resulting degree of divergence from time inconsistency, we next introduce an error measure based on the loss in welfare associated with following sophisticated versus resolute choice.

Based on eqns. (14)-(17) and (20)-(25), respectively, let U^R and U^S denote, respectively, the utility or "welfare" associated with resolute and sophisticated demands, where resolute demand is derived purely based on the period one utility function and sophisticated choice is derived following backward induction. The utility of both the optimal resolute and sophisticated consumption trees are computed based on the consumer's period one DOCE utility (33) given below.[28] Then the error measure is defined as

$$err = \left(\frac{U^R}{U^S}\right)^{-\frac{1}{\delta_1}} - 1, \tag{32}$$

where the power $-1/\delta_1$ is used to normalize the utility function into a homogeneous function of degree 1. As mentioned above, when TC holds, resolute and sophisticated choice will converge and the error measure becomes zero. Therefore, it is natural to use it to measure the level of divergence from TC. When this measure becomes small (e.g., the DOCE consumer's welfare losses are one percent or less), we will say that the consumer's preferences are almost time consistent and TC almost holds. (We follow similar terminology introduced by Tirole (2002, p. 645) in his Presidential address on "Rational Irrationality" when considering time inconsistent preferences.)[29]

We first show that for the case of three time periods, if the DOCE CES time preference utility takes the special Cobb-Douglas form, an infinitesimal departure from ICER results in identical "welfare" associated with DOCE resolute and

[28] Since in general sophisticated choice will no longer be rationalizable by a well-defined utility function, we use the consumer's resolute utility to determine the welfare loss that results from time inconsistent choices.

[29] Although our application is not exactly the same, like in Tirole (2002), we consider a setting in which time inconsistency arises and are concerned with the divergence in utility values from the case where preferences are time consistent.

sophisticated demands. Then second, for discrete departures from ICER, numerical analyses verify that the resulting welfare differences associated with TC not holding exactly can be surprisingly small. Moreover, the assumed restriction on CES time preferences can be relaxed over a range of EIS-values and the welfare differences can still remain small. DOCE preferences can be almost time consistent even for a 90 year time horizon (see Figure 6).

5.1 *Almost Time Consistent DOCE Preferences*

In Section 3, we showed that if a consumer's preferences over consumption trees are represented by DOCE utility where the time preference and risk preference building block utilities, respectively, take the CES and CRRA forms and asset returns characterizing the consumption trees satisfy ICER, the preferences satisfy TC. That is, the consumer's preferred consumption trees based on sophisticated and resolute choice are identical, and hence the utility values of the optimal demands, or "welfare", are the same for the sophisticated and resolute solution techniques. Here we consider the case where ICER is relaxed and CES-CRRA preferences continue to be represented by

$$U = \begin{cases} -\frac{c_1^{-\delta_1}}{\delta_1} - \sum_{t=2}^{T} \beta^{t-1} \frac{\widehat{c}_t^{-\delta_1}}{\delta_1} & (\delta_1 > -1, \delta_1 \neq 0) \\ \ln c_1 + \sum_{t=2}^{T} \beta^{t-1} \ln \widehat{c}_t & (\delta_1 = 0) \end{cases}, \tag{33}$$

where

$$\widehat{c}_t = \left(E\left[\widetilde{c}_t^{-\delta_2} \right] \right)^{-\frac{1}{\delta_2}}.$$

Asset markets are assumed to be complete and departures from ICER are evaluated in terms of changes in asset returns. The following result considers an infinitesimal change in asset returns from the case where $ICER$ holds. Then if the DOCE CES building block utility takes the special Cobb-Douglas form where the $EIS = 1$, the resulting changes in U^S and U^R are equal.

PROPOSITION 3. *Assume that the consumer has DOCE utility corresponding to the CES-CRRA form (33) and chooses over three period consumption trees and asset* *markets are complete. If $EIS = 1$, an infinitesimal change in returns results in the welfare of resolute and sophisticated choice satisfying*

$$\left.\frac{\partial U^S}{\partial R(s^t)}\right|_{\mathbf{R}} = \left.\frac{\partial U^R}{\partial R(s^t)}\right|_{\mathbf{R}} \quad \textit{and} \quad \left.\frac{\partial U^S}{\partial R_f(s^t)}\right|_{\mathbf{R}} = \left.\frac{\partial U^R}{\partial R_f(s^t)}\right|_{\mathbf{R}}$$

when evaluated at returns $\mathbf{R}$ that satisfy ICER.[30]

In the proof of Proposition 3, it will prove convenient to assume that the change in asset returns in our complete market setting is a consequence of a change in the contingent claim prices. Such a change will, in general, result in a change in both the risky and risk free asset returns. The proposition states that any infinitesimal change in asset returns from their values when ICER holds results in no welfare loss to a sophisticated DOCE consumer.

REMARK 4. Despite the EIS being a central parameter in modern dynamic macro and finance models, no clear consensus exists on its numerical value. However, the interesting, recent study of Crump, et al. (2022) estimates the EIS to be between 0.5 and 0.8 with 0.5 being their best estimate. They suggest that the latter value is consistent with much of the literature. This study utilizes a unique data set based on individual consumer subjective expectations. Their analysis replaces the more traditional approach where data is based on realizations. Also, it should be emphasized that it is based on individual consumer demand behavior consistent with the analysis in this paper rather than equilibrium models calibrations which in many cases report much higher estimates of the EIS. Our EIS assumption in Proposition 3 lies a bit above the upper bound of the Crump, et al. (2022) EIS-values. DOCE preferences can be almost time consistent for smaller values of the EIS (such as 0.5 or 0.25).

[30] Since conditional demands are the same for naive and sophisticated choice at the prices where ICER holds, one can follow a similar argument to verify that the derivatives with respect to the asset returns are also the same for naive and resolute choice.

REMARK 5. It is easy to see that DOCE preferences are indifferent to correlation of consumption over time. That is, the consumer is indifferent between two consumption trees that are identical in consumption values except that consumption at different nodes for one tree are independent over time and for the other tree they are correlated. This is not the case for the EZW preference model. However, if we compare the i.i.d. asset returns and correlated asset returns (with the same return values), DOCE preferences will generate different optimal demands as will EZW preferences. It is important to note that the implausibly large timing premia referenced in Epstein, Farhi and Strzalecki (2014, p. 2683) is associated with the high persistence of the equilibrium consumption growth process (autocorrelation) assumed for instance in the long run risk asset pricing model of Bansal and Yaron (2004). For the case of the zero persistence associated with i.i.d. consumption growth, the timing premia are lower but not zero. It should be emphasized that this analysis is based on an asset pricing equilibrium model and not the micro demand analysis assumed in this paper as evidenced by the equilibrium assumptions being made on an exogenous consumption growth process whereas in our analysis consumption growth is endogenous as a consequence of assumptions related to asset returns. If one assumes that asset returns are i.i.d. as in the special case of ICER assumed in prior sections of this paper, consumption growth will also be i.i.d. However, for the discrete perturbations of asset returns considered in the next subsection, autocorrelated asset returns can be associated with a persistent consumption growth process.[31]

5.2 *A Numerical Robustness Analysis*

In order to apply standard recursive methods, we assume that uncertainty is Markovian. Suppose time is $t = 1, \dots, T$. Shocks, s_t realize in a finite set $\{1, \dots, S\}$ and follow a Markov chain with transition π. As in Subsection 2.1, we denote a node of the event-tree by s^t since it can be identified by a history of shocks up to

[31]We thank a referee for stressing the important linkages between the timing premia in the equilibrium analysis of Epstein, Farhi and Strzalecki (2014) and the demand analysis considered in this paper.

some time t. Asset returns at each node s^t, $t = 2, \dots, T$ can then be written as

$$\mathbf{R}(s^t) = \mathbf{R}(s_t | s_{t-1}).$$

That is to say, asset returns at all nodes can be summarized by S^2 vectors $\mathbf{R}(s'|s) \in \mathbb{R}^J_+$.

The agent's maximization problem is described in eqns. (10) - (13) and resolute and sophisticated choice are defined by eqns. (14)–(17) and (20)-(25). We define U^S to be the utility the agent derives from optimal sophisticated choice and U^R to be the utility derived from optimal resolute choice. We consider deviations from ICER and report welfare losses from sophisticated choice relative to resolute choice. We report these in consumption equivalent terms based on the error measure (32) defined above.

Throughout we assume that there are two shocks. We first fix the time horizon, $T = 30$, as well as preferences, $\delta_1 = 1$, $\delta_2 = 4$, $\beta = 1$ and assume that shocks are i.i.d., i.e., $\pi(s|s') = 0.5$ for all s, s'. In this setting, the ICER assumption is satisfied if $\mathbf{R}(s'|1) = \mathbf{R}(s'|2)$ for all s'. We assume that markets are complete, i.e., there are two assets. Without loss of generality we can take the first asset to be risk free, i.e., $R_1(s'|s) = R_f(s)$ for all s, s'. Also without loss of generality we can take the second asset to return zero if the current shock is two, i.e., $R_2(2|s) = 0$ for $s = 1, 2$.[32]

One important deviation from ICER arises when the risk free rate is stochastic, i.e., $R_f(1) \neq R_f(2)$. We assume that

$$R_f(1) = 1 - \Delta_i, \quad R_f(2) = 1 + \Delta_i,$$

and compute the effects of Δ_i on our error, err. By the absence of arbitrage, eqn. (9), we must have [33]

$$\frac{1}{R_2(1|s)} = \lambda_s \frac{1}{R_f(s)}, \ \lambda_s \in (0,1) \text{ for } s = 1, 2.$$

[32]Supplemental Appendix C provides a summary of the numerical simulation methodology employed in our analysis.

[33]Note that the three parameters Δ_i, λ_1 and λ_2 then pin down all returns on the entire tree where the return of the risk free asset only depends on the previous node and the return of the risky asset on the up-branch depends on the previous node and on the down-branch is always zero.

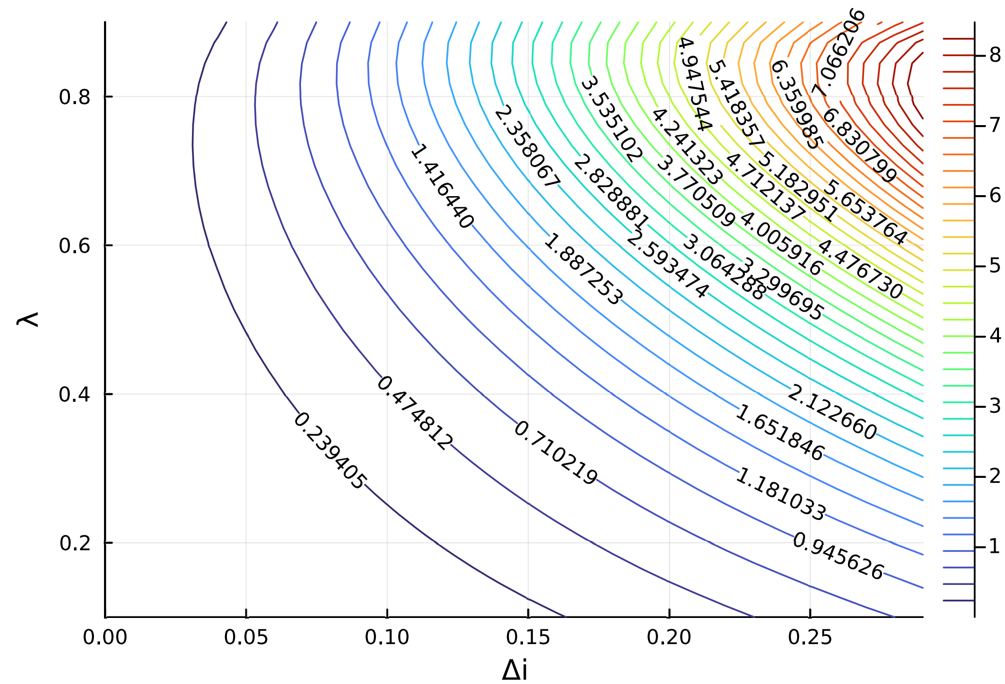


FIGURE 5. Percentage Error for Different Δ_i,λ

For a given Δ_i we can then conduct a systematic search of values of λ_1 and λ_2 that maximize err. We assume that $\lambda_1, \lambda_2 \in [0.1, 0.9]$.

Figure 5 depicts a contour plot of percentage errors for different combinations of Δ_i and $\lambda = \lambda_1$.[34] For each (Δ_i, λ), the figure shows the largest error that can be obtained for any $\lambda_2 \in [0.1, 0.9]$.

In this setting, ICER never holds even when $\Delta_i = 0$ because the risky returns are different for the two states. However, as can be seen in the figure, errors are very small, in fact they turn out to lie below 0.23 percent. For example, if $R_f(1) = 0.9$, $R_f(2) = 1.1$, i.e., if the (net) risk free rate varies between -10 and 10 percent, it is clear that ICER must be violated. If, in addition $R_2(1|1) = 2.25$, (i.e., if $\Delta_i = 0.1$ and $\lambda = 0.4$) the error is below 0.47 percent independent of the risky return in state 2.

[34] In Figure 5, as well as Figures **??** and **??**, a color scale is provided on the right-hand side to indicate the size of the errors.

As the risk free rate across the two shocks becomes more volatile, errors increase and can reach about 8 percent when $R_f(1) = 0.7, R_f(2) = 1.3$. Remarkably, errors stay extremely low as long as the difference in risk free rates across the two states is below 0.2 ($\Delta_i = 0.1$). For a number of historical periods, this is not an unrealistic range for variations in the real rate. It is also interesting to note that in the low interest rate state ($s = 1$), a high return for the risky asset that pays only in the low interest rate state leads to the lowest errors (whenever $\lambda = 0.1$ errors are low across different values of Δ_i). Our computational results indicate that for the high interest rate state ($s = 2$), the opposite holds. The error-maximizing λ_2 is typically around 0.1. In the following we will take $\lambda_1 = 0.9$ and $\lambda_2 = 0.1$. While these are typically not the values that maximize the error, they always close to the error-maximizing values.

5.2.1 *Robustness in T, Discounting, Persistence, EIS and RRA* As above, we consider two shocks $s = 1, 2$ and fix $\delta_1 = 1$, $\delta_2 = 4$. In order to depict how errors change as T, β and the persistence of the shock change,[35] it turns out to be useful to only vary Δ_i and report errors for $\lambda_1 = 0.9$ and $\lambda_2 = 0.1$. As explained above, this is close to error-maximizing.

We first examine the effect of the number of time periods on errors. For $\beta = 1$ and $\pi(s|s') = 0.5$ we vary T to take the values $5, 15, 30, 60$ as well as 90.

Figure 6 shows how the error depends on the length of the agent's planning horizon. The error increases substantially as one moves from 5 to 15 periods. This seems somewhat intuitive as the time inconsistency problem might become more severe as the planning horizon becomes larger. However, if one considers $T = 30$, $T = 60$ and $T = 90$, errors increase by much less. In particular, the difference between $T = 60$ and $T = 90$ seems to suggest that errors stabilize and eventually do not increase anymore with the time horizon.

We next fix $T = 30$, and examine the effects of β and the persistence of the exogenous shock on errors. For the left panel, we assume $\pi(s|s') = 1/2$ and for the right panel we take $\beta = 1$.

[35] In our symmetric setup, we define the persistence of a Markovian shock as the probability of staying in the same shock.

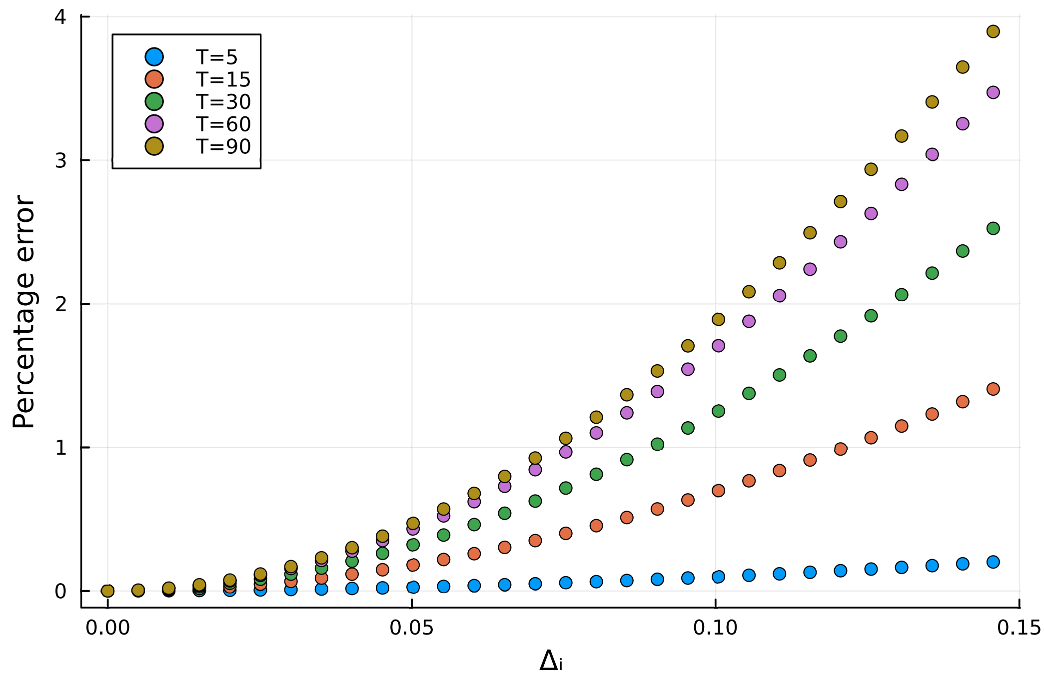
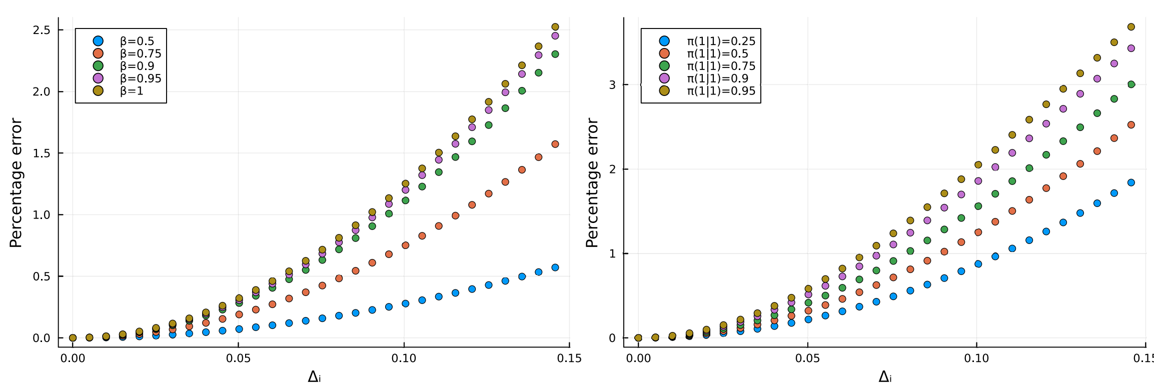


FIGURE 6. Error for Different T



(a) (b)

FIGURE 7. Effects of Different β (Left Panel) and Different Persistence (Right Panel)

As can be seen in Figure 7, higher discounting (lower β) decreases the average error slightly, but the effect is rather small. Higher persistence of the exogenous shock increases the error – again, the effects are rather small.

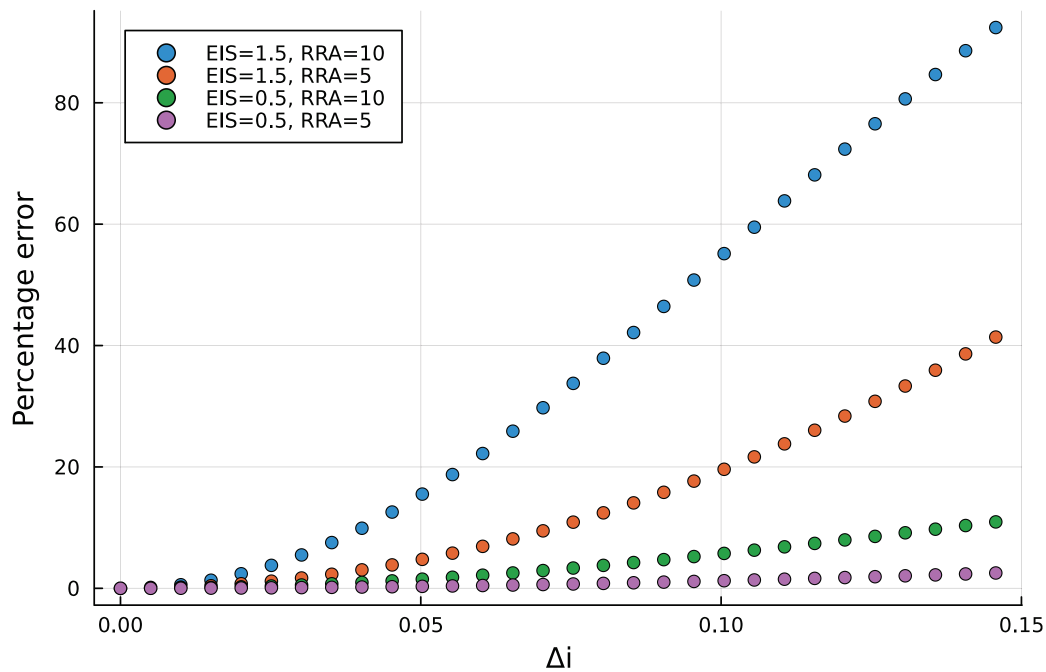


FIGURE 8. Error for Different EIS and RRA

Finally, we consider how the percentage errors corresponding to different values of Δ_i are affected by different combinations of the time preference δ_1 and risk preference δ_2 parameters. To facilitate comparisons with the analysis in the next subsection, we consider the corresponding values of EIS and RRA.

In Figure 8, we assume the EIS equals 0.5 and 1.5, where, respectively, the former value is suggested in the analysis by Crump, et al. (2022) (referenced in Remark 4 above) and the latter is considered in long run risk asset pricing models (e.g., Bansal and Yaron (2004)). We also consider RRA values of 5 and 10, where the latter is employed in Mehra and Prescott (1985) and Bansal and Yaron (2004) and the former value is a more moderate level of risk aversion. The figure demonstrates that for $EIS = 0.5$, the percentage error is relatively small even as Δ_i increases. When $EIS = 1.5$, the errors become quite large. The figure also indicates that the error can be quite sensitive to an increase in the RRA when the higher value of EIS is assumed.

For asset demand and saving applications, it is important to evaluate whether the specification of the price process (characterized, for instance, by our assumed values of Δ_i and λ) assumed in this section are realistic. In the next subsection, we revisit the crucial role played by the assumed (EIS, RRA) on the size of the welfare loss associated with TC not holding exactly.

5.2.2 *Revisiting the Respective Roles of* EIS *and* RRA To continue employing the simple 2-state Markov environment considered in the prior subsection, we follow the analysis in Melino and Yang (2003)[36]. They construct a 2 by 2 stochastic discount factor that can match the historical first two moments of the equity return and the risk free rate. They assume a Markov transition matrix

$$\pi = \begin{bmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{bmatrix}.$$

From this and from the stochastic discount factor in Melino and Yang (2003), we can construct $R_f(1) = 1.0641$ and $R_f(2) = 0.9519$ (so Δ_i is around 0.06 in this calibration) and $\lambda_1 = 0.3884$ and $\lambda_2 = 0.1480$. It is clear from our analysis above that the resulting returns, while not satisfying ICER, do not lead to the largest errors. Since Melino and Yang (2003) calibrate to yearly data, we take $T = 60$ and $\beta = 0.96$.

Figure 9 shows the resulting errors for a range of EIS and RRA values. Corresponding to different (RRA, EIS)-combinations, we plot multiple level curves each of which is characterized by constant percentage errors or welfare losses associated TC not holding exactly. It should be noted that the lowest plotted level curve is associated with the very small error of less than $2/3$ of 1 percent. The dot corresponding to $RRA = 2$ and $EIS = 0.5$ corresponds to the case where $\delta_1 = \delta_2 = 1$ and the DOCE consumer's preferences are characterized by the EU special case and resolute and sophisticated choice are the same. As a result, TC

[36]The simulations in this subsection are based on the data in Melino and Yang (2003), since to our knowledge it is the only published paper that calibrates a finite Markov chain of Arrow prices. We acknowledge that our assumption of a 2 state process is overly simplistic, but it allows us to use the Melino and Yang (2003) calibration, which is a bit of a benchmark.

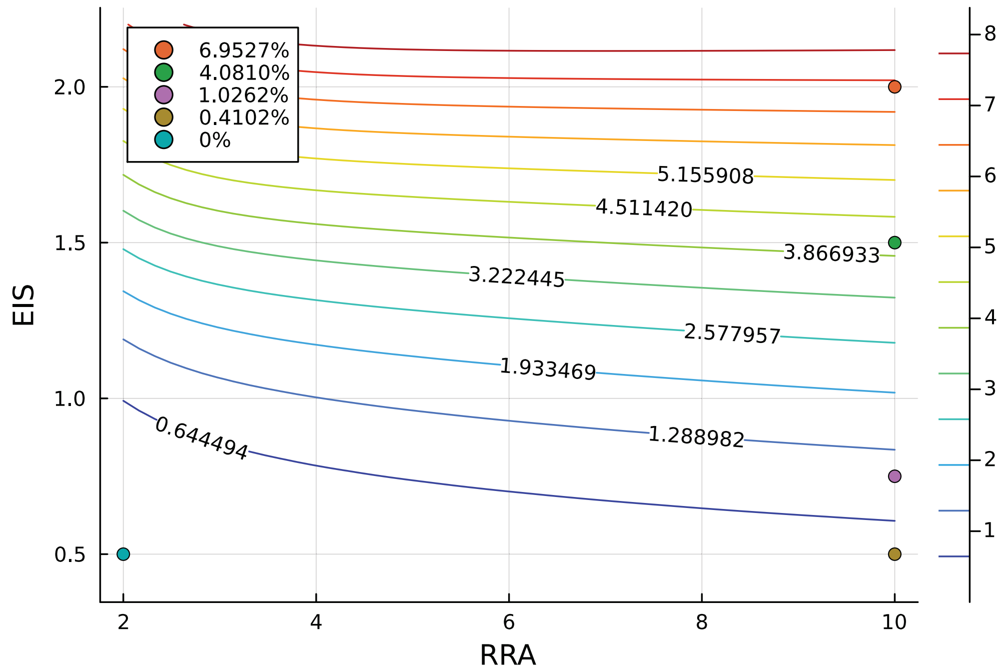


FIGURE 9. Error Level Curves: Different (RRA, EIS)

holds and the error equals 0. Indeed, one can imagine a partial level curve passing through this point associated with different RRA and EIS combinations in the positive orthant where the EU case holds. Given that a number of asset pricing analyses, such as Bansal and Yaron (2004), focus on the case where $RRA = 10$, or higher, we consider four special cases with this value of RRA and different values of the EIS. These points correspond, respectively, in Figure 9 to an EIS of 0.5, 0.75, 1.5 and 2.0 and are identified by the four dots associated with $RRA = 10$. The welfare losses of departures from TC not holding exactly for these four points is given in the box in the northwest corner of the figure. It can be seen that for the range of EIS recommended in Crump, et al. (2022), the errors are one percent or smaller. For the case of the EIS and RRA values assumed in the long run risk literature, the errors do rise, but only to about 7%. These simulation results suggest that perhaps the deviations from TC holding exactly might not be terribly concerning for the consumption and asset demand optimization of consumers following sophisticated choice especially if the EIS is below one, as is often assumed in demand analyses. For the case of long run risk models where the $EIS = 1.5$ and $RRA = 10$, it would seem highly desirable to extend the analysis in this paper to the analysis of equilibrium asset prices based on the assumption of a DOCE representative agent following sophisticated choice. In particular, is the resulting equity risk premium materially smaller than for the case of EZ preferences? This indeed would be consistent with a smaller timing premium and with DOCE preferences satisfying TRI.

6. Concluding Comments

In this paper, we provide conditions that ensure that CES-CRRA DOCE preferences exhibit TC, SEP and TRI on a restricted domain of consumption trees corresponding to the optimal solution of the consumption-portfolio problem. Under these conditions, it is possible to extend the classic Fisherian consumption saving analysis to a setting with risky and risk free investment opportunities. The same conditions also imply that if the KP time and risk preference building block utilities are the same as for the DOCE preferences, the KP and DOCE optimal demands become identical. Although the assumption that asset returns are ICER or i.i.d. is quite strong, it can be viewed as the cost of being able to derive conditions on time and risk preference parameters in DOCE saving and portfolio demand comparative statics independent of temporal resolution preferences. We also identify conditions under which CES-CRRA DOCE preferences can continue to be almost time consistent when considering quantitatively reasonable departures from ICER.

Two potential extensions of the analysis in this paper would seem to be of interest. As we have shown, when the assumption of i.i.d. or ICER asset returns does not hold, EZ and DOCE preferences generate different optimal demands.

When the conditions discussed in the prior section are satisfied and the divergence from TC holding can be viewed as small, it would seem worthwhile to explore the feasibility of deriving a DOCE equilibrium model. For example, is it possible to construct a long run risk model based on CES-CRRA DOCE preferences where the representative agent follows sophisticated choice with desirable properties such as a zero timing premium to resolve all of the risk in the second time period? [37] For the case where the welfare loss due to divergence between resolute and sophisticated demands becomes material, it would also seem to be of interest to explore how to trade off that loss with the difference in timing premia between EZ and DOCE equilibria.

The second relates to the observation in Section 1 that in intertemporal demand problems the presence of TC behavior does not depend just on preferences, but prices (or asset returns) can also play a crucial role. This differs from typical decision theoretic analyses such as in KP and Johnsen and Donaldson (1985), where the conditions for preferences to satisfy TC are implicitly assumed to hold for all prices. Our key condition ICER has been shown essentially to be a restriction on return distributions or, for complete markets, on contingent claim prices. It would be interesting to consider more generally when such cases can arise. Consider the variation of HARA preferences in Theorem A.1 in Supplemental Appendix A where U takes the CES form and V takes the CARA form. Our result does not extend to this case. However, it can be verified that for a simplified tree structure and under additional restrictions including $\beta R_{f3} = 1$, a consumer with DOCE preferences becomes time consistent. It is interesting to note that this particular combination of time and risk preferences is assumed in Weil (1993) and more recently in variations of Hansen and Sargent (1995) preferences such as Tallarini (2000). Collectively, these results suggest the potential value of future research into the general question of joint restrictions on preferences and prices such that dynamic choice behavior is time consistent.

[37] To the extent that the equilibrium model is characterized by autocorrelated consumption growth, the interrelations between temporal resolution and correlation preferences raised by Stanca (2023) may become relevant.

Appendix A: Proof of Proposition 1

Generalizing the example in Subsection 3.1 and denoting by $\alpha(s^t)=c(s^t)/c(s^{t-1})$,

$$\mathcal{U}(\alpha|s^\tau)=u(c(s^\tau))+\beta u\circ V^{-1}\left(\sum_{s^{\tau+1}}\pi(s^{\tau+1}|s^\tau)V(\alpha(s^{\tau+1})c(s^\tau))\right)+\ldots+$$
$$\beta^{T-1}u\circ V^{-1}\left(\begin{array}{c}\sum_{s^{\tau+1}\succ s^\tau}\pi(s^{\tau+1}|s^\tau)\ldots\sum_{s^T\succ s^\tau}\pi(s^T|s^{T-1})\\ V(\alpha(s^{\tau+1})\cdot\ldots\cdot\alpha(s^T)c(s^\tau))\end{array}\right)$$
$$=u(c(s^\tau))+\beta u\circ V^{-1}\left(\sum_{s^{\tau+1}\succ s^\tau}\pi(s^{\tau+1}|s^\tau)V(\alpha(s^{\tau+1})c(s^\tau))\right)K_{\tau+1},$$

where $K_{\tau+1}$ is recursively defined as

$$K_T=1+\beta u\circ V^{-1}\left(\sum_{s^T\succ s^{T-1}}\pi(s^T|s^{T-1})V(\alpha(s^T))\right)$$

and

$$K_t=1+\beta u\circ V^{-1}\left(\sum_{s^t\succ s^{t-1}}\pi(s^t|s^{t-1})V(\alpha(s^t))\right)K_{t+1}$$

for $t=\tau+1,\ldots T-1$. Note that $\mathbf{c}\in\mathcal{I}$ ensures that K_t does not depend on s^t.

By the same argument as in the example, it is now clear that if α is preferred to $\widetilde{\alpha}$ at τ it must be preferred at $\tau-1$ and, by induction, preferred at any $\tau-i$, $i=1,\ldots,\tau-1$.

Appendix B: Proof of Theorem 1

In order to facilitate the proof, we first introduce Assumption ICER* and show that it is equivalent to ICER. Define recursively for each s^t, $t=T-1,T-2,\ldots$, $\widehat{\mathbf{n}}(s^t)\in\mathbb{R}^J$ to be the unique solution to the J equations

$$\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)R(s^{t+1})V'\left(\frac{R(s^{t+1})\cdot\widehat{\mathbf{n}}(s^t)}{1+\sum_j\widehat{\mathbf{n}}_j(s^{t+1})}\right)=$$

$$\frac{1}{\beta(u\circ V^{-1})'\left(\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)V\left(\frac{R(s^{t+1})\cdot\widehat{\mathbf{n}}(s^t)}{1+\sum_j\widehat{\mathbf{n}}_j(s^{t+1})}\right)\right)},$$

where $\widehat{\mathbf{n}}(s^t)=\mathbf{n}(s^t)/c(s^t)$ and $\widehat{\mathbf{n}}(s^T)=0$ for all s^T.

Assumption ICER* Assume that for all s^t, $t<T$,

$$\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)V\left(\frac{R(s^{t+1})\cdot\widehat{\mathbf{n}}(s^t)}{1+\sum_j\widehat{\mathbf{n}}_j(s^{t+1})}\right)=K_t,$$

where K_t only depends on t.

To show the equivalence of ICER* and ICER, it suffices to show that under ICER, $\widetilde{n}_j(s^t)$ (as defined in eqn. (30)) is constant across all s^t for given t. By induction, we consider first $t=T-1$. Homotheticity ensures that ICER can be written as

$$\sum_{s^{t+1}\succ s^t}\pi(s^{t+1}|s^t)R(s^{t+1})\cdot\widetilde{\mathbf{n}}(s^t)V'\left(R(s^{t+1})\cdot\widetilde{\mathbf{n}}(s^t)\right)=\tilde{K}_t,$$

which is independent of s^t. Taking the J equations in (30) and weighting each j with $\widetilde{n}_j(s^t)$ and summing up, this implies that $\sum_j\widetilde{n}_j(s^t)$ must be independent of s^t, for $t=T-1$. But then the same argument applies for each $t<T$ and ICER and ICER* are equivalent conditions.

Next, we prove that ICER* together with homothetic utility is sufficient. The following first order conditions are necessary and sufficient for optimization at s^τ for consumption at some future node $\bar{s}^t\succeq s^\tau$

$$V'(c(\bar{s}^t))\left(u\circ V^{-1}\right)'\left(\sum_{s^t\succ s^\tau}\pi(s^t|s^\tau)V(c(s^t))\right)=$$

$$\beta(u\circ V^{-1})'\left(\sum_{s^{t+1}\succ s^\tau}\pi(s^{t+1}|s^\tau)V\left(c(s^{t+1})\right)\right)$$

$$\sum_{s^{t+1}\succ \bar{s}^t} R(s^{t+1})\pi(s^{t+1}|\bar{s}^t)V'\left(c\left(s^{t+1}\right)\right), \tag{34}$$

for all $\bar{s}^t$, $t<T$. Since $u(.)$ and $V(.)$ are assumed to be homothetic, it is clear that these necessary and sufficient first order conditions will be satisfied for some $\alpha(s^t)$ that satisfy ICER*, and that these $\alpha(s^t)$ do not change with τ. Therefore the choice does not change with τ and choices are time consistent.

To prove necessity of homothetic utility given ICER*, consider the simplified case of three periods, $t=1,2,3$, based on a version of the consumption tree depicted in Figure 1 where there are just two branches. Suppose markets are complete. To satisfy Assumption ICER* suppose that the prices of the contingent claims are identical and denoted by $p(2)$.

The first order conditions for optimal choice at $t=2$ are $p(2)u'(c_{2s})=\beta u'(c_{3s})$ $(s=1,2)$ and, at $t=1$, planning for $t=2$, are

$$p(2)V'(c_{2s})(u\circ V^{-1})'\left(\sum_s \pi_s V(c_{2s})\right)=\beta(u\circ V^{-1})'\left(\sum_s \pi_s V(c_{3s})\right)V'(c_{3s}).$$

The first equation implies $c_{3s}=u'^{-1}\left(p(2)u'(c_{2s})/\beta\right)$ and substituting this into the second equation, we obtain

$$\begin{aligned} &p(1)V'(c_{2s})(u\circ V^{-1})'\left(\sum_s \pi_s V(c_{2s})\right)= \\ &\beta(u\circ V^{-1})'\left(\sum_s \pi_s V\left(u'^{-1}\left(\frac{p(1)}{\beta}u'(c_{2s})\right)\right)\right) \\ &V'\left(u'^{-1}\left(\frac{p(2)}{\beta}u'(c_{2s})\right)\right). \end{aligned} \tag{35}$$

Denote the price $p(2)$ simply by p. Then we consider variations in $p(2)=p$ as well as first period prices $p(1)$ that keep second period consumption fixed. Taking the derivative with respect to p on both sides and then setting $p=\beta$, one obtains

$$1=\frac{(u\circ V^{-1})''\left(\sum_s \pi_s V(c_{2s})\right)\sum_s \pi_s\left(V'(c_{2s})u'^{-1\prime}\circ u'(c_{2s})u'(c_{2s})\right)}{(u\circ V^{-1})'\left(\sum_s \pi_s V(c_{2s})\right)} + \frac{V''(u'^{-1\prime}u'(c_{2s}))u'(c_{2s})}{V'(c_{2s})}.$$

Taking derivatives with respect to c_{2s}, $s=1,2$, we obtain[38]

$$\frac{d}{dc}\frac{f'^{-1\prime}(g(c))g(c)}{f(c)}=0,$$

where $f(c)=V'(c)$ and $g(c)=u'(c)$. Since $g^{-1\prime}(g(c))g'(c)=1$, we obtain

$$\frac{d}{dc}\frac{f'(c)g(c)}{g'(c)f(c)}=0.$$

Consider the following ordinary differential equation

$$\frac{d}{dc}\left(\frac{f'(c)\,g(c)}{f(c)\,g'(c)}\right)=0.$$

We have

$$\frac{f'(c)\,g(c)}{f(c)\,g'(c)}=K_1,$$

where K_1 is a constant. Therefore,

$$\frac{f'(c)}{f(c)}=(\ln f(c))'=K_1\frac{g'(c)}{g(c)}=K_1(\ln g(c))',$$

implying that $\ln f(c)=K_1\ln g(c)+K_2$, where K_2 is a constant. Thus we have $f(c)=K_3(g(c))^{K_1}$, where K_3 is a constant.

[38] This is possible since we can vary the prices of both contingent claims at $t=1$ independently.

Assuming $K > 0$, we can write $V'(c) = u'^K$ and $V'^{-1}(x) = u'^{-1}(x^{1/K})$. Substituting this into (35), we obtain

$$p(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right) = \beta(u \circ V^{-1})' \left(\sum_s \pi_s V \left(u'^{-1} \left(\frac{p}{\beta} u'(c_{2s}) \right) \right) \right) \left(\frac{p}{\beta} \right)^{\frac{1}{K}}.$$

Since $u \circ V^{-1}(x) = x^\nu$ for some ν, it follows that the above can only hold if $u\left((u')^{-1}(x)\right)$ is homothetic. In this case, we can write $u\left((u')^{-1}(x)\right) = ax^\delta$. Then we have $(u')^{-1}(x) = u^{-1}\left(ax^\delta\right)$. Assuming $(u')^{-1}(x) = y$, then

$$u^{-1}\left(ax^\delta\right) = y \Leftrightarrow x = \left(\frac{u(y)}{a}\right)^{\frac{1}{\delta}}.$$

Therefore, we have $u'(x) = a(u(x))^\delta$. Thus if $\delta \neq 1$, we have

$$\frac{d(u(x))^{1-\delta}}{dx} = a(1-\delta) \Rightarrow u(x) = (a(1-\delta)x + c)^{-\frac{1}{1-\delta}}.$$

This corresponds to the DARA or IARA case of the HARA class. If $\delta = 1$,

$$\frac{d \ln u(x)}{dx} = a(1-\delta) \Rightarrow u(x) = \exp(a(1-\delta)x + c).$$

A simple numerical example can show that DARA and IARA utilities within the HARA class do not produce time consistent demand unless the conditions of Theorem A.1 in Supplemental Appendix A hold.

In the last step we prove that under homothetic utility, the assumption ICER* is necessary for time consistency. Suppose ICER* does not hold and consider the first order conditions for optimal choice at some date τ of assets at some future $t > \tau$

$$V'(c(s^t))\left(u \circ V^{-1}\right)' \left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau) V(c(s^t)) \right) =$$

$$\beta(u \circ V^{-1})' \left(\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau) V\left(c(s^{t+1})\right) \right) \sum_{s^{t+1} \succ s^t} R(s^{t+1}) \pi(s^{t+1}|s^t) V'\left(c\left(s^{t+1}\right)\right).$$

It is clear that they can only be satisfied for the same choices $c(s^t), c(s^{t+1}),\ s^{t+1} \succ s^t$, at two different dates τ, τ' if the ratio of the terms in the large parentheses are the same, i.e., if

$$\frac{\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau) V(c(s^t))}{\sum_{s^{t+1} \succ s^\tau} \pi(s^{t+1}|s^\tau) V\left(c(s^{t+1})\right)}$$

is independent of τ. But this implies that $\mathbf{c} \in \mathcal{I}$ which can only hold if ICER* or equivalently ICER holds. This completes the proof.

Appendix C: Proof of Proposition 2

The first key insight is that DOCE and KP preferences generate identical utility functions over $\mathcal{I}$. To see this, let $\alpha(s^t) = c(s^t)/c(s^{t-1})$ and recall that DOCE utility can be written as follows

$$\begin{aligned}
\mathcal{U}(\alpha|s^\tau) &= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) V(\alpha(s^{\tau+1})c(s^\tau))\right) + \ldots + \\
&\quad \beta^{T-1} u \circ V^{-1}\left(\begin{array}{c} \sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) \ldots \sum_{s^T \succ s^{T-1}} \pi\left(s^T|s^{T-1}\right) \\ V(\alpha(s^{\tau+1}) \cdot \ldots \cdot \alpha(s^T) c(s^\tau)) \end{array}\right) \\
&= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s^{\tau+1} \succ s^\tau} \pi\left(s^{\tau+1}|s^\tau\right) V(\alpha(s^{\tau+1})c(s^\tau))\right) K_{\tau+1},
\end{aligned}$$

where $K_{\tau+1}$ is recursively defined as

$$K_T = 1 + \beta u \circ V^{-1}\left(\sum_{s^T \succ s^{T-1}} \pi\left(s^T|s^{T-1}\right) V(\alpha(s^T))\right)$$

and

$$K_t = 1 + \beta u \circ V^{-1}\left(\sum_{s^t \succ s^{t-1}} \pi\left(s^t|s^{t-1}\right) V(\alpha(s^t))\right) K_{t+1}.$$

Similarly KP utility can be written as

$$\begin{aligned}
&\mathcal{U}^{KP}(\mathbf{c}|s^\tau) \\
&= -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \frac{\beta\left(\sum_{s^{\tau+1}\succ s^\tau} \pi(s^{\tau+1}|s^\tau)\mathcal{U}^{KP}(\mathbf{c}|s^{\tau+1})^{-\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} \\
&= -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \beta\frac{c(s^\tau)^{-\delta_1}}{\delta_1} \\
&\quad \left(\begin{array}{c} \left(\sum_{s^{\tau+1}\succ s^\tau} \pi(s^{\tau+1}|s^\tau))\alpha(s^{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\ \left(1+\beta\left(\sum_{s^{\tau+2}\succ s^{\tau+1}} \pi(s^{\tau+2}|s^{\tau+1})\alpha(s^{\tau+2})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}(1+\ldots)\right)\end{array}\right),
\end{aligned}$$

which, when multiplied out, is identical to DOCE utility. It remains to be shown that optimal choice under KP utility lies in $\mathcal{I}$. The necessary and sufficient conditions for optimal choice can be written as

$$\begin{aligned}
u'(c(s^t)) = \beta(u\circ V^{-1})'\left(\sum_{s^{t+1}\succ s^t} V\circ\mathcal{U}(\mathbf{c}|s^{t+1})\right) \\
\sum_{s^{t+1}\succ s^t} \pi(s^{t+1}|s^t)R(s^{t+1})(V\circ u^{-1})'\mathcal{U}(\mathbf{c}|s^{t+1})u'(c(s^{t+1})).
\end{aligned}$$

At $T-1$ KP and DOCE coincide, hence we can substitute for $\mathcal{U}$ and we obtain that $\sum_{s^{T-1}\succ s^{T-2}} \pi(s^{T-1}|s^{T-2})V(\alpha(s^{T-1}))$ is constant for all s^{T-2}. By induction this is then true for all t and hence KP and DOCE preferences generate the same demands.

Appendix D: Proof of Proposition 3

Since markets are complete, one can equivalently solve the dynamic choice problem in the contingent claim setting. Any infinitesimal perturbation in returns is equivalent to an infinitesimal perturbation to contingent claim prices. Suppose we assume the case of one risky and one risk free asset, then

$$p_{31} = \frac{\xi_{f31} p - \xi_{32} p_f}{(\xi_{31} - \xi_{32})\xi_{f31}} \quad \text{and} \quad p_{32} = \frac{\xi_{31} p_f - \xi_{f31} p}{(\xi_{31} - \xi_{32})\xi_{f31}},$$

and if we only change p_{31} and keep p_{32} fixed, in general both asset prices and returns will change. Without loss of generality, consider a four-branch tree and assume that only the price p_{31} for good c_{31} changes.[39] When ICER holds, we have shown that $(c_1, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, c_{34})$ are the same for resolute and sophisticated choice. For the four-branch tree resolute choice, the budget constraint in period one is

$$c_1 + p_{21}\left(c_{21} + p_{31}c_{31} + p_{32}c_{32}\right) + p_{22}\left(c_{22} + p_{33}c_{33} + p_{34}c_{34}\right) = I.$$

Thus we can define the resolute indirect utility as

$$W^R(p, I) = \max_{\mathbf{c} \geq 0} U(\mathbf{c}) \;\; s.t.$$

$$c_1 + p_{21}\left(c_{21} + p_{31}c_{31} + p_{32}c_{32}\right) + p_{22}\left(c_{22} + p_{33}c_{33} + p_{34}c_{34}\right) = I.$$

By the envelope theorem,

$$\frac{\partial W^R}{\partial p_{31}} = \lambda^R p_{21} c_{31},$$

where λ^R is the Lagrange multiplier. The first-order conditions (which are necessary and sufficient in this setting) imply

$$\frac{\partial U}{\partial c_{31}} = \beta^2 \widehat{c}_3^{\delta_2} \pi_{31} c_{31}^{-1-\delta_2} = \lambda^R p_{21} p_{31},$$

where

$$\widehat{c}_3 = \left(\pi_{21}\left(\pi_{31}c_{31}^{-\delta_2} + \pi_{32}c_{32}^{-\delta_2}\right) + \pi_{22}\left(\pi_{33}c_{31}^{-\delta_2} + \pi_{34}c_{32}^{-\delta_2}\right)\right)^{-\frac{1}{\delta_2}}.$$

[39] For the tree with more than four branches, the following calculations can be trivially generalized. The final expression for $\left(\partial U^S/\partial p_{31}\right) / \left(\partial U^R/\partial p_{31}\right)$ is the same as the four-branch tree case except that the definition for $\widehat{c}_3$ requires appropriate modification.

Therefore, we have

$$\frac{\partial W^R}{\partial p_{31}} = \frac{\beta^2 \widehat{c}_3^{\delta_2} \pi_{31} c_{31}^{-1-\delta_2} p_{21} c_{31}}{p_{21} p_{31}} = \frac{\beta^2 \pi_{31} \widehat{c}_3^{\delta_2} c_{31}^{-\delta_2}}{p_{31}}.$$

For sophisticated choice, in period two, for the upper branch, the indirect utility function at the 21 node is

$$W^{S21} = \max_{(c_{21}, c_{31}, c_{32}) \geq 0} \ln c_{21} - \frac{\beta \ln \left(\pi_{31} c_{31}^{-\delta_2} + \pi_{32} c_{32}^{-\delta_2} \right)}{\delta_2} s.t.$$

$$c_{21} + p_{31} c_{31} + p_{32} c_{32} = I_{21}$$

The envelope theorem implies

$$\frac{\partial W^{S21}}{\partial p_{31}} = \lambda^S c_{31}.$$

The first order conditions for optimal choice at node (21) imply

$$\frac{\partial U}{\partial c_{31}} = \beta^2 \widehat{c}_{31}^{\delta_2} \pi_{31} c_{31}^{-1-\delta_2} = \lambda^S p_{31},$$

where

$$\widehat{c}_{31} = \left(\pi_{31} c_{31}^{-\delta_2} + \pi_{32} c_{32}^{-\delta_2} \right)^{-\frac{1}{\delta_2}}.$$

Since we assume $\ln$ time preference utility, optimal resolute consumption at node (21) will remain unchanged. By the envelope theorem any changes in optimal sophisticated choice in period one will have no effect on welfare. Therefore, if we define sophisticated indirect utility as $W^S(p, I)$, we have

$$\frac{\partial W^S(p, I)}{\partial p_{31}} = \frac{\partial W^{S21}(p, I)}{\partial p_{31}} \frac{\widehat{c}_3^{\delta_2}}{\widehat{c}_{31}^{\delta_2}}.$$

We obtain

$$\frac{\partial W^S}{\partial p_{31}} = \frac{\beta^2 \pi_{31} \widehat{c}_3^{\delta_2} c_{31}^{-\delta_2}}{p_{31}} = \frac{\partial W^R}{\partial p_{31}}.$$

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