

Supplement to “Optimal contracts with a risk-taking agent”

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These proofs refer to results in the text and the Appendices.

D.4 Proof of existence for $\underline{u} = -\infty$

Suppose that $\underline{u} = -\infty$, which corresponds to the agent having no limited liability constraint. This section gives conditions under which a unique solution to (P) exists and satisfies certain properties. Say that $u(\cdot)$ is *regular* if $\underline{w} = -\infty$ and $\frac{u'(w)u''(w)}{[u''(w)]^2} < 3$ for all $w \in \mathbb{R}$. These conditions are quite mild; in particular, the second condition means that $u'(\cdot)$ is not excessively convex, in the sense that it has local concavity everywhere greater than -2 . See [Prékopa \(1973\)](#) and [Borell \(1975\)](#) for details.

PROPOSITION 10. *Suppose $\pi(y) \equiv y$, $u(\cdot)$ is strictly concave and regular, and $\underline{u} = -\infty$. Then for any $a \geq 0$, there exists a unique contract $v(\cdot)$ that implements a at maximum profit. Furthermore, there exists $\bar{u} < \infty$ independent of $\underline{u}$ such that $v(\bar{y}) < \bar{u}$ and $v(\underline{y}) > -\bar{u}$.*

PROOF. Given Lemma 6, it is enough to show that for some $\underline{u}$, $v_{\underline{u}}(\underline{y}) > \underline{u}$. Assume not, so that, in particular, for all $\underline{u}$, $v_{\underline{u}}(\underline{y}) = \underline{u}$. We show that this leads to a contradiction. We henceforth restrict attention to $\underline{u} \leq 0$. For $\underline{u}$ sufficiently negative, it cannot be the case that $v_{\underline{u}}$ is linear. In particular, if $v_{\underline{u}}$ is linear, then since $v_{\underline{u}}(\bar{y}) > u_0 + c(a)$, we have that

$$\int v_{\underline{u}}(x) f_a(x|a) dx = \int v'_{\underline{u}}(x) (-F_a(x|a)) dx \geq \frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}},$$

which diverges in $\underline{u}$, contradicting that $v_{\underline{u}}$ must satisfy (IC-FOC) with equality. Hence, for each $\underline{u}$, we can take a point $x_{\underline{u}} \in C_{v_{\underline{u}}}$, and derive $\lambda_{\underline{u}}$ and $\mu_{\underline{u}}$ as in the proof of Proposition 3.

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Let $z_{\underline{u}}(\cdot) = \rho(\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a))$, where we follow the convention that $\rho(s) = -\infty$ for $s \leq 0$. The contract $v_{\underline{u}}$ will, in general, differ from $z_{\underline{u}}$, since $z_{\underline{u}}$ need be neither concave nor satisfy the limited liability constraint. Note that $n_{\underline{u}}(\cdot) = \rho^{-1}(v_{\underline{u}}(\cdot)) - (\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a)) \stackrel{s}{=} v_{\underline{u}}(\cdot) - z_{\underline{u}}(\cdot)$.

STEP 1. There is $\bar{\mu} < \infty$ such that $\mu_{\underline{u}} \leq \bar{\mu}$ for all $\underline{u}$.

PROOF. Applying a small positive amount of $t_{x_{\underline{u}}, \bar{y}}$ adds cost at rate at most $\rho^{-1}(\bar{u}) \times \int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a) dx$, adds incentives at rate $\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a) dx$, and relaxes (IR). It follows that

$$\mu_{\underline{u}} \leq \rho^{-1}(\bar{u}) \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a) dx}.$$

But, as in the proof that $|Q(\mathbf{0})| > 0$,

$$\begin{aligned} \frac{\partial}{\partial x_{\underline{u}}} \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a) dx} &= - \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a) dx} + \frac{\int_{x_{\underline{u}}}^{\bar{y}} f_a(x|a) dx}{\int_{x_{\underline{u}}}^{\bar{y}} f(x|a) dx} \\ &\leq 0, \end{aligned}$$

and so we can take

$$\bar{\mu} = \rho^{-1}(\bar{u}) \frac{\int (x - \underline{y}) f(x|a) dx}{\int (x - \underline{y}) f_a(x|a) dx} < \infty. \quad \triangleleft$$

STEP 2. There is $\underline{\mu} > 0$ and $\underline{u}^* > -\infty$ such that $\mu_{\underline{u}} \geq \underline{\mu}$ for all $\underline{u} < \underline{u}^*$.

PROOF. Choose $-\infty < \underline{u}^* \leq 0$ such that

$$\rho^{-1}(\underline{u}^*) < \frac{1}{2} \rho^{-1}(u_0 + c(a)), \quad (15)$$

$$c'(a) < \frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}}, \quad (16)$$

where such a $\underline{u}^*$ exists since by assumption $\lim_{w \rightarrow -\infty} \frac{1}{u'(w)} = 0$. Let

$$r \equiv \sup_{\tau \in [\frac{1}{2} \rho^{-1}(u_0 + c(a)), \infty)} \rho'(\tau).$$

Since $\rho(1/u'(w)) = u(w)$, we have that

$$\rho'\left(\frac{1}{u'(w)}\right) = \frac{(u')^3}{-u''}(w),$$

from which

$$\frac{\rho''\left(\frac{1}{u'(w)}\right)}{\rho'\left(\frac{1}{u'(w)}\right)} = u'(w) \left(\frac{u'''(w)u'(w)}{(u''(w))^2} - 3 \right). \quad (17)$$

Since u is regular, it follows that $\rho'' < 0$ and so $r < \infty$. Let $\bar{l}_x = \max_x l_x(x|a)$ and choose $\underline{\mu} > 0$ such that

$$\underline{\mu} < \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)}, \quad (18)$$

$$\underline{\mu} < \frac{1}{r\bar{l}_x} \frac{u_0 + c(a)}{\bar{y} - \underline{y}}. \quad (19)$$

Assume that for some $\underline{u} < \underline{u}^*$, $\mu_{\underline{u}} < \underline{\mu}$. We show that this leads to a contradiction, establishing the result.

Using Corollary 2 (which depends only on the necessity part of the proof of Proposition 3, which is proved in Appendix B) and the fact that $\bar{y}$ is free, $n(\bar{y}) \leq 0$, and so $\lambda_{\underline{u}} + \mu_{\underline{u}}l(\bar{y}|a) \geq \rho^{-1}(v_{\underline{u}}(\bar{y})) \geq \rho^{-1}(u_0 + c(a))$. Thus,

$$\begin{aligned} \lambda_{\underline{u}} + \mu_{\underline{u}}l(\underline{y}|a) &= \lambda_{\underline{u}} + \mu_{\underline{u}}l(\bar{y}|a) - \mu_{\underline{u}}(l(\bar{y}|a) - l(\underline{y}|a)) \\ &\geq \rho^{-1}(u_0 + c(a)) - \frac{1}{2} \frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)} (l(\bar{y}|a) - l(\underline{y}|a)) \\ &= \frac{1}{2} \rho^{-1}(u_0 + c(a)), \end{aligned} \quad (20)$$

where the inequality follows from $\mu_{\underline{u}} < \underline{\mu}$ and (18).

Since $\underline{u} < \underline{u}^*$, and by (15), $\rho^{-1}(v_{\underline{u}}(\underline{y})) = \rho^{-1}(\underline{u}) < \frac{1}{2} \rho^{-1}(u_0 + c(a))$. Thus, using (20), $n(\underline{y})$ is strictly positive and it follows by Corollary 2 that $v_{\underline{u}}$ begins with a linear segment, the slope of which (by concavity) is at least

$$\frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}} \geq \frac{u_0 + c(a)}{\bar{y} - \underline{y}}.$$

But using (20) and the definition of r , we have that for all x ,

$$\begin{aligned} z'_{\underline{u}}(x) &= \rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}l(x|a))\mu_{\underline{u}}l_x(x|a) \\ &\leq r\mu_{\underline{u}}\bar{l}_x \\ &< \frac{u_0 + c(a)}{\bar{y} - \underline{y}}, \end{aligned}$$

where the strict inequality follows from (19). Hence, the initial linear segment of $v_{\underline{u}}$ crosses $z_{\underline{u}}$ at most once (from below). This implies that the entire contract is, in fact, linear with slope at least $(u_0 + c(a) - \underline{u})/(\bar{y} - \underline{y})$. In particular, let x_H be the right end

of the linear segment. If x_H is at or before the crossing point, then $v_{\underline{u}}$ violates (2) and so cannot be optimal by part (i) of Proposition 3. If $x_H < \bar{y}$ is after the crossing, then we violate Corollary 2. It follows that $v_{\underline{u}}$ generates incentives at least

$$\frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}} > c'(a)$$

using (16). But we have shown that (IC-FOC) binds at $v_{\underline{u}}$, leading to the desired contradiction. $\triangleleft$

STEP 3. There is $u_0 + c(a) > u_c > -\infty$ such that if $\underline{u} < \underline{u}^*$ and $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) < u_c$, then $z_{\underline{u}}(\cdot)$ is concave at x .

PROOF. Note first that ρ is trivially concave anywhere that it is equal to $-\infty$ and that, by assumption, $\lim_{s \rightarrow 0} \rho(s) = -\infty$. Hence, it is enough to prove concavity where $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)))$ is finite. But it follows from (17) and the fact that u is regular that $\lim_{t \downarrow 0} \rho''(t)/\rho'(t) = -\infty$ and so $\rho''(t)/\rho'(t)$ is negative for t below some t' . Assume $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) < t'$. Then

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) &= \frac{\partial}{\partial x} (\rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} l_x(x|a)) \\ &= \rho''(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) (\mu_{\underline{u}} l_x(x|a))^2 \\ &\quad + \rho'(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} l_{xx}(x|a) \\ &= \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \mu_{\underline{u}} + \frac{l_{xx}}{l_x^2}(x|a) \\ &\leq \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) \underline{\mu} + \frac{l_{xx}}{l_x^2}(x|a). \end{aligned}$$

The second term is bounded by assumption. The first term diverges to $-\infty$ as $\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)) \rightarrow 0$. Hence, since ρ is monotone and since $\lim_{w \rightarrow -\infty} u'(w) = \infty$, the result follows. $\triangleleft$

STEP 4. As in the derivation of r in Step 2, let $\hat{r}$ be such that for all $t \geq \rho^{-1}(u_c)$, $\rho'(t) \leq \hat{r}$. Let $-\infty < \hat{\underline{u}} \leq \underline{u}^*$ satisfy

$$\hat{s} \equiv \frac{u_0 + c(a) - \hat{\underline{u}}}{\bar{y} - \underline{y}} \geq \max\{c'(a), \bar{\mu} \bar{l}_x \hat{r}\}$$

and assume that $\underline{u} < \hat{\underline{u}}$. Then $z_{\underline{u}}(\underline{y}) \leq \underline{u}$.

PROOF. Assume that $z_{\underline{u}}(\underline{y}) > \underline{u}$. Then, since $v_{\underline{u}}(\underline{y}) = \underline{u}$, $v_{\underline{u}}$ begins with a linear segment of positive length of slope at least $\hat{s}$, and so by Proposition 3 and part (i) of Definition 2, crosses $z_{\underline{u}}$ from below and is strictly above $z_{\underline{u}}$ for an interval of positive length as well. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. If $v_{\underline{u}}$ has its initial crossing of $z_{\underline{u}}$ at or before $x_{\underline{u},c}$, then

since $z_{\underline{u}}$ is concave until $x_{\underline{u},c}$, $v_{\underline{u}}$ remains above $z_{\underline{u}}$ until $x_{\underline{u},c}$. But then, since for $x > x_{\underline{u},c}$, $\hat{s} \geq z'_{\underline{u}}$, $v_{\underline{u}}$ in fact never re-crosses $z_{\underline{u}}$. Alternatively, if the initial crossing of $z_{\underline{u}}$ by $v_{\underline{u}}$ is after $x_{\underline{u},c}$, then again, since $v_{\underline{u}}$ has slope greater than $z'_{\underline{u}}$ for $x > x_{\underline{u},c}$, $v_{\underline{u}}$ never re-crosses $z_{\underline{u}}$. In either case, by Corollary 2, $v_{\underline{u}}$ is thus linear on all of $[\underline{y}, \bar{y}]$, a contradiction. $\triangleleft$

STEP 5. Let $u_{y_0} = u_0 + c(a) - c'(a)(\bar{y} - y_0) > -\infty$. Then $v_{\underline{u}}(y_0) \geq u_{y_0}$.

PROOF. Since $v_{\underline{u}}(\bar{y}) \geq u_0 + c(a)$, it follows that everywhere on $[\underline{y}, y_0]$, $v_{\underline{u}}(\cdot)$ is below the line $L(\cdot)$ that goes through $(y_0, v_{\underline{u}}(y_0))$ and $(\bar{y}, u_0 + c(a))$, and everywhere on $(y_0, \bar{y}]$, $v_{\underline{u}}(\cdot)$ is above $L(\cdot)$. Hence, since $f_a < 0$ on $[\underline{y}, y_0]$ and $f_a > 0$ on $(y_0, \bar{y}]$,

$$\begin{aligned} c'(a) &= \int v_{\underline{u}}(x) f_a(x|a) dx \\ &\geq \int L(x) f_a(x|a) dx \\ &= \frac{u_0 + c(a) - v_{\underline{u}}(y_0)}{\bar{y} - y_0}. \end{aligned}$$

Rearranging yields the desired result. $\triangleleft$

STEP 6. Choose $\infty < u_s < \min\{u_{y_0}, u_c, \rho(-\bar{\mu}l(\underline{y}|a)), \hat{u}\}$ small enough that for all $t \leq u_s$,

$$\rho'\left(\frac{1}{u'(u^{-1}(t))}\right) \bar{\mu}l_x \geq \hat{s}, \quad (21)$$

where $l_x = \min_x l_x(x|a) > 0$. Since $\rho'(\tau)$ diverges to ∞ as $\tau \downarrow 0$, and since $1/u'(u^{-1}(t))$ goes to 0 as $t \downarrow -\infty$, such a u_s is guaranteed to exist.

STEP 7. Choose $\underline{u} < u_s$. Let $\bar{z}_{\underline{u}}(\cdot) = \rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\cdot|a))$, where $\bar{\lambda}_{\underline{u}}$ solves $\rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\underline{y}|a)) = \underline{u}$. By Step 4, $z_{\underline{u}}(\underline{y}) \leq \underline{u}$, and so, since $\mu_{\underline{u}} \leq \bar{\mu}$, $z_{\underline{u}}(\cdot) \leq \bar{z}_{\underline{u}}(\cdot)$. Let $x_{\underline{u},s}$ be defined by $\bar{z}_{\underline{u}}(x_{\underline{u},s}) = u_s$. Since $\bar{\lambda}_{\underline{u}} + \bar{\mu}l(\underline{y}|a) = 1/u'(u^{-1}(\underline{u})) > 0$, it follows that $\bar{\lambda}_{\underline{u}} + \bar{\mu}l(y_0|a) \geq -\bar{\mu}l(\underline{y}|a)$ and, hence,

$$\rho(\bar{\lambda}_{\underline{u}} + \bar{\mu}l(y_0|a)) > \rho(-\bar{\mu}l(\underline{y}|a)) > u_s,$$

where the last inequality is by definition of u_s in Step 6. Thus, $x_{\underline{u},s} < y_0$.

STEP 8. For all $x < x_{\underline{u},s}$, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$.

PROOF. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. By construction, $\bar{z}_{\underline{u}}(\cdot)$ is concave where $x \leq x_{\underline{u},c}$. Using (21), $\bar{z}'_{\underline{u}}(\cdot) > \hat{s}$ for $x < x_{\underline{u},s}$ and $\bar{z}'_{\underline{u}}(\cdot) < \hat{s}$ for $x \geq x_{\underline{u},c}$. Assume that for some $\tilde{x} < x_{\underline{u},s}$, $v_{\underline{u}}(\tilde{x}) > \bar{z}_{\underline{u}}(\tilde{x}) \geq z_{\underline{u}}(\tilde{x})$. By Corollary 2, $v_{\underline{u}}$ is linear at $\tilde{x}$. If $v'_{\underline{u}}(\tilde{x}) \leq \bar{z}'_{\underline{u}}(\tilde{x})$, then, since $\bar{z}_{\underline{u}}$ is concave on $[\underline{y}, x_{\underline{u},s}]$ and again using Corollary 2, $v_{\underline{u}}$ is also above $\bar{z}_{\underline{u}}$ and, hence, is linear, for all x in $[\underline{y}, \tilde{x}]$. But then

$$v_{\underline{u}}(\underline{y}) - \bar{z}_{\underline{u}}(\underline{y}) \geq v_{\underline{u}}(\tilde{x}) - \bar{z}_{\underline{u}}(\tilde{x}) > 0,$$

contradicting that $v_{\underline{u}}(\underline{y}) = \underline{u}$. Thus, $v'_{\underline{u}}(\tilde{x}) > \bar{z}'_{\underline{u}}(\tilde{x}) > \hat{s}$. But then $v_{\underline{u}}$ remains linear and, hence, strictly above the concave function $z_{\underline{u}}$ at least until $x_{\underline{u},c}$. For $x \geq x_{\underline{u},c}$, $\bar{z}'_{\underline{u}}(\tilde{x}) \leq \hat{s}$, and so as before v can never re-cross $\bar{z}_{\underline{u}}$, and so a fortiori can never re-cross $z_{\underline{u}}$. Hence, $v_{\underline{u}}$ is linear on $[\tilde{x}, \bar{y}]$, with slope at least $\hat{s}$. Let L be the line that agrees with $v_{\underline{u}}$ on $[\tilde{x}, \bar{y}]$. To the left of $\tilde{x}$, $v_{\underline{u}}$, being concave, lies below L . But $\tilde{x} < x_{\underline{u},s} < y_0$ and so, since $f_a(\cdot|a)$ is negative on $[\underline{y}, \tilde{x}]$,

$$\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int L(x) f_a(x|a) dx \geq \hat{s} > c'(a),$$

again a contradiction. $\triangleleft$

STEP 9. We show that $\lim_{\underline{u} \rightarrow -\infty} \int v_{\underline{u}}(x) f_a(x|a) dx = \infty$. For $\underline{u}$ sufficiently negative, this provides the necessary contradiction to the original supposition that $v_{\underline{u}}(\underline{y}) = \underline{u}$ for all $\underline{u}$, proving the result.

PROOF. By Step 8, for $\underline{u}$ sufficiently negative, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$ for all $x \leq x_{\underline{u},s}$. Let $v_{\underline{u}}^T$ truncate $v_{\underline{u}}$ to never pay more than u_s . Since $\max(0, v_{\underline{u}}(x) - u_s)$ is an increasing function, $\int \max(0, v_{\underline{u}}(x) - u_s) f_a(x|a) dx \geq 0$ and, hence, $\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int v_{\underline{u}}^T(x) f_a(x|a) dx$. Note also that since $v_{\underline{u}}(y_0) > u_s$, $v_{\underline{u}}^T(x) = u_s$ for all $x \geq y_0$. Let $\bar{z}_{\underline{u}}^T$ similarly truncate $\bar{z}_{\underline{u}}$ to pay u_s to the right of $x_{\underline{u},s}$. Then $\bar{z}_{\underline{u}}^T$ is everywhere at least as large as $v_{\underline{u}}^T$, but equal to $v_{\underline{u}}^T$ everywhere to right of y_0 . Hence, since f_a is negative to the left of y_0 , we have

$$\int v_{\underline{u}}(x) f_a(x|a) dx \geq \int v_{\underline{u}}^T(x) f_a(x|a) dx \geq \int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx.$$

To arrive at a contradiction, it would thus be enough to show that $\int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx$ diverges as $\underline{u} \rightarrow -\infty$. But by Moroni and Swinkels (2014, Lemma 4), under our regularity conditions, $\int \bar{z}_{\underline{u}}(x) f_a(x|a) dx$ does diverge as $\underline{u} \rightarrow -\infty$.

Let

$$u^{**} = \rho(1 + \bar{\mu}(l(\bar{y}|a) - l(y|a))) < \infty.$$

Then, for all $\underline{u}$ sufficiently negative that $\frac{1}{u'(u^{-1}(\underline{u}))} \leq 1$, $\bar{z}_{\underline{u}}(\bar{y}) \leq u^{**}$. Hence,

$$\begin{aligned} \int \bar{z}_{\underline{u}}(x) f_a(x|a) dx - \int \bar{z}_{\underline{u}}^T(x) f_a(x|a) dx &= \int (\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)) f_a(x|a) dx \\ &\leq \int_{y_0}^{\bar{y}} (\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)) f_a(x|a) dx \\ &\leq (u^{**} - u_s) \int_{y_0}^{\bar{y}} f_a(x|a) dx \\ &< \infty, \end{aligned}$$

where the first inequality follows because $\bar{z}_{\underline{u}}(x) - \bar{z}_{\underline{u}}^T(x)$ is weakly positive, and the second inequality follows because it is bounded above by $u^{**} - u_s$. $\square$

D.5 Agent reports x

In this section, we allow the agent to send a contractible message $\tilde{x}$ after he observes x but before y is realized. Payments can therefore depend on both $\tilde{x}$ and y , which allows the principal to discipline the agent from engaging in risk-taking. Restricting attention to the case where both parties are risk-neutral, we show that a linear contract is optimal in this setting.

Since the principal does not benefit from risk-taking, it is without loss to restrict attention to mechanisms that punish the agent as much as possible whenever his report does not match the final output: $s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}$ for some upper semicontinuous function $s(\cdot)$. Then the principal's problem is

$$\begin{aligned} \max_{a, s(\cdot)} \quad & \mathbb{E}_{F(\cdot|a), G} [y - s(y)\mathbb{I}_{\{y=\tilde{x}\}} + M\mathbb{I}_{\{y\neq\tilde{x}\}}] \\ \text{subject to} \quad & a, G, \tilde{x} \in \arg \max_{\tilde{a}, \tilde{G} \in \mathcal{G}, \tilde{x}} \{ \mathbb{E}_{F(\cdot|\tilde{a}), \tilde{G}} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] - c(\tilde{a}) \}, \\ & \mathbb{E}_{F(\cdot|a), G} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] - c(a) \geq u_0, \\ & s(\cdot) \geq -M, \end{aligned}$$

where $\tilde{x}$ maps x to a report made to the principal.

Fix $s(\cdot)$, and consider the agent's choice of G_x and $\tilde{x}$ following any intermediate output $x > \underline{y}$. Define

$$\lambda_s(x) = \max \{ \lambda : \lambda(y - \underline{y}) - M = s(y) \text{ for some } y \geq x \}.$$

Intuitively, $\lambda_s(x)$ is the smallest slope such that $\lambda_s(x)(y - \underline{y}) - M \geq s(y)$ for all $y \geq x$. We show that following intermediate output $x > \underline{y}$, the agent optimally chooses G_x and $\tilde{x}$ so that his expected payoff is $\lambda_s(x)(x - \underline{y}) - M$.²⁵

LEMMA 7. *For any $s(\cdot)$ and $x \in \mathcal{Y}$, the principal's expected payment to the agent equals*

$$\sigma_s(x) \equiv \max_{G_x, \tilde{x}} \{ \mathbb{E}_{G_x} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] \} = \begin{cases} s(\underline{y}) & \text{if } x = \underline{y}, \\ \lambda_s(x)(x - \underline{y}) - M & \text{if } x > \underline{y}. \end{cases} \quad (22)$$

PROOF. Fix $s(\cdot)$ and $x > \underline{y}$. First, we show that there exists some G_x and $\tilde{x}$ such that $\mathbb{E}_{G_x} [s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}] = \lambda_s(x)(x - \underline{y}) - M$. By definition of $\lambda_s(\cdot)$, there exists a $\hat{y} \geq x$ such that $\lambda_s(x)(\hat{y} - \underline{y}) - M = s(\hat{y})$. Let $\tilde{x} = \hat{y}$ and $G_x(y) = (1 - p_{\hat{y}}) + p_{\hat{y}}\mathbb{I}_{\{y \geq \hat{y}\}}$, where $p_{\hat{y}} = \frac{x - \underline{y}}{\hat{y} - \underline{y}}$; i.e., $y = \underline{y}$ with probability $1 - p_{\hat{y}}$ and $y = \hat{y}$ with probability $p_{\hat{y}}$. Then the agent's expected payoff is

$$\begin{aligned} p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}s(\hat{y}) - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \frac{x - \underline{y}}{\hat{y} - \underline{y}}[\lambda_s(x)(\hat{y} - \underline{y}) - M] - \frac{\hat{y} - x}{\hat{y} - \underline{y}}M \\ &= \lambda_s(x)(x - \underline{y}) - M. \end{aligned}$$

²⁵If $x = \underline{y}$, then the agent is compelled to choose $G_{\underline{y}}(y) = 1$, so his expected payoff is equal to $s(\underline{y})$.

Next we show that the agent cannot earn more than $\lambda_s(x)(x - \underline{y}) - M$ following intermediate output x . For any report $\tilde{x}$, the agent earns more than $-M$ only if $y = \tilde{x}$, so his optimal distribution G_x maximizes the probability that $y = \tilde{x}$ subject to the constraint that $\mathbb{E}_{G_x}[y] = x$. This is accomplished by choosing $G_x(\cdot)$ such that $y = \tilde{x}$ with some probability $p_{\tilde{x}}$ and $y = \underline{y}$ with probability $1 - p_{\tilde{x}}$, where $p_{\tilde{x}}\tilde{x} + (1 - p_{\tilde{x}})\underline{y} = x$. It suffices to show that the agent's expected payoff under this distribution is maximized if $\tilde{x} = \hat{y}$.

Suppose that there exists some $\tilde{x} \neq \hat{y}$ such that $p_{\tilde{x}}s(\tilde{x}) - (1 - p_{\tilde{x}})M > p_{\hat{y}}s(\hat{y}) - (1 - p_{\hat{y}})M = \lambda_s(x)(x - \underline{y}) - M$. Then there must exist some $\tilde{\lambda} > \lambda_s(x)$ such that $\tilde{\lambda}(\tilde{x} - \underline{y}) - M = s(\tilde{x})$, which contradicts the definition of $\lambda_s(x)$. Therefore, for all x , the agent's expected payoff equals $\lambda_s(x)(x - \underline{y}) - M$. $\square$

To see this result, recall that the agent earns $-M$ whenever his report does not equal the realized output. Therefore, if he misreports $\tilde{x} \neq x$, then he chooses G_x to maximize the probability that $y = \tilde{x}$. In particular, it is optimal for G_x to put weight on only two points, $\tilde{x}$ and $\underline{y}$. Given this $\tilde{x}$, the agent's payoff can be written as $p_{\tilde{x}}s(\tilde{x}) - (1 - p_{\tilde{x}})M$, where $p_{\tilde{x}}\tilde{x} + (1 - p_{\tilde{x}})\underline{y} = x$. It can be shown that the agent's payoff can be rewritten as $\lambda(x - \underline{y}) - M$, where $\lambda \leq \lambda_s(x)$. There exists some report $\tilde{x}$ that sets $\lambda = \lambda_s(x)$, proving the result.

Using Lemma 7, we can rewrite the principal's problem as

$$\begin{aligned} \max_{a, s(\cdot)} \quad & \mathbb{E}_{F(\cdot|a)}[x - \sigma_s(x)] \\ \text{subject to} \quad & a \in \arg \max_{\tilde{a}} \{ \mathbb{E}_{F(\cdot|\tilde{a})}[\sigma_s(x)] - c(\tilde{a}) \}, \\ & \mathbb{E}_{F(\cdot|a)}[\sigma_s(x)] - c(a) \geq u_0, \\ & s(\cdot) \geq -M, \end{aligned}$$

where, for any contract $s(\cdot)$, $\sigma_s(\cdot)$ is given by (22).

Recall the definition of $s_a^L(\cdot)$ from Section 4. We show that if $a \geq 0$ is such that (LL) holds with equality after $\underline{y}$ under $s_a^L(\cdot)$, then $s_a^L(\cdot)$ implements a at maximum profit in this setting. Consequently, if (LL) binds for the optimal $a \geq 0$, then a linear contract is optimal as in Proposition 2.

PROPOSITION 11. *Fix any effort $a \geq 0$. If $s_a^L(\underline{y}) = -M$, then $s_a^L(\cdot)$ implements a at maximum profit.*

PROOF. Note that $\lambda_s(\cdot)$ is decreasing for any $s(\cdot)$ and, moreover, is constant for all $x \in \mathcal{Y}$ if $s(\cdot)$ is affine. Let $\hat{s}(\cdot)$ implement a at maximum profit and suppose there exists $x_L < x_H$ such that $\lambda_{\hat{s}}(x_L) > \lambda_{\hat{s}}(x_H)$.

Define $s_L(y) = \beta(y - \underline{y}) - M$, where β is chosen such that $\mathbb{E}_{F(\cdot|a)}[s_L(y) - \lambda_{\hat{s}}(y) \times (y - \underline{y}) + M] = 0$. Such a β exists by the intermediate value theorem because $\lambda_{\hat{s}}(y) \geq 0$ is finite. Since $\lambda_{\hat{s}}(\cdot)$ is strictly decreasing over some interval, there exists some $y^* \in (\underline{y}, \bar{y})$ such that $\lambda_{\hat{s}}(y) \geq \beta$ if and only if $y \leq y^*$. Then $\beta - \lambda_{\hat{s}}(y)$ is first negative and then

positive, $\int [\beta - \lambda_{\hat{s}}(y)](y - \underline{y})f(y|a) dy = 0$ by construction, and $\frac{f_a(\cdot|a)}{f(\cdot|a)}$ is strictly increasing, so Beesack's inequality implies that

$$\int [\beta - \lambda_{\hat{s}}(y)](y - \underline{y})f_a(y|a) dy > 0.$$

Therefore, $s_L(\cdot)$ implements some effort level $a' > a$, which implies that $\beta > c'(a)$.

Observe that $s_a^L(y) < s_L(y)$ for all $y > \underline{y}$, because $s_a^L(y) = -M$ by assumption and $c'(a) < \beta$. Moreover, $s_a^L(\cdot)$ implements a and satisfies both the individual rationality and the limited liability constraints. Therefore, $s_a^L(\cdot)$ implements effort a at strictly higher profit than $\hat{s}(\cdot)$. So $\lambda_{\hat{s}}(\cdot)$ must be constant and $\sigma_{\hat{s}}(\underline{y}) = -M$, in which case $s_a^L(\cdot)$ is also optimal. $\square$

D.6 Comparative static of optimal contract with respect to $\underline{y}$

This appendix considers how a^* changes with the lower bound $\underline{y}$ on output. A decrease in $\underline{y}$ implies that the agent can take on more severe left-tail risk by gambling over worse outcomes. We prove that a lower $\underline{y}$ makes it costlier for the principal to induce any nonzero effort level. As $\underline{y}$ approaches $-\infty$, inducing any positive effort becomes arbitrarily expensive and so the agent exerts no effort in the optimal contract.

COROLLARY 3. *Consider a decreasing sequence $\{\underline{y}_k\}_{k=0}^\infty$ with $\lim_{k \rightarrow \infty} \underline{y}_k = -\infty$. For each $k \geq 0$, consider $\mathcal{Y} = [\underline{y}_k, \bar{y}]$ and some output distribution $F_k(\cdot|a)$ that satisfies our assumptions (i.e., has full support on $[\underline{y}_k, \bar{y}]$, satisfies $\mathbb{E}_{F_k(\cdot|a)}[x] = a$, etc.), and let a_k^* be the corresponding optimal effort. Then $\lim_{k \rightarrow \infty} a_k^* = 0$, and if $\pi(y) \equiv y$, then a_k^* is decreasing in k .*

Proposition 2 implies that the principal's expected payment from inducing $a^* \geq 0$ equals $E_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + w)]$. For small enough $\underline{y}$, $s_{a^*}^L(y) = -M$. But then implementing $a^* > 0$ becomes arbitrarily costly as $\underline{y} \rightarrow -\infty$, in which case the principal is better off not motivating the agent at all. If the principal is risk-neutral, then we can show that the principal's profit under $s_{a^*}^L(\cdot)$ is supermodular in a^* and $\underline{y}$, so that a^* is increasing in $\underline{y}$.

PROOF OF COROLLARY 3. Fix $\hat{a} > 0$. Define

$$y_1 \equiv \min_{a \in [\hat{a}, a^{\text{FB}}]} \left\{ a - \frac{c(a) + u_0 + M}{c'(a)} \right\}$$

and

$$y_2 \equiv \min_{a \in [\hat{a}, a^{\text{FB}}]} \left\{ \frac{u^{-1}(u_0) - (1 - c'(a)a) - M}{c'(\hat{a})} \right\},$$

and note that since $c'(a) \geq c'(\hat{a}) > 0$ for all $a \geq \hat{a}$, $y_{\min} \equiv \min\{0, y_1, y_2\} > -\infty$.

Let $\underline{y} < y_{\min}$ and suppose toward a contradiction that there exists a distribution $F(\cdot|a)$ on $[\underline{y}, \bar{y}]$ such that effort $a^* \geq \hat{a}$ is optimal under $F(\cdot|a)$. Note first that Proposition 2 implies that the principal's expected payoff equals

$$\mathbb{E}_{F(\cdot|a^*)}[\pi(y - s_{a^*}^L(y))] = \mathbb{E}_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + \min\{M, c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0\})].$$

Since $\underline{y} < y_1$, $c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0 > M$. Furthermore, the principal's payoff is bounded above by

$$\pi((1 - c'(a^*))a^* + c'(a^*)\underline{y} + M)$$

by Jensen's inequality. Since $\underline{y} < \min\{0, y_2\}$, $(1 - c'(a))a + c'(a)\underline{y} + M < u^{-1}(u_0)$ for any $a \in [\hat{a}, a^{\text{FB}}]$. But then $a^* \geq \hat{a}$ cannot be optimal because it is strictly dominated by $a^* = 0$ and $s(\cdot) \equiv u^{-1}(u_0)$, a contradiction. Hence, for $\underline{y} < y_{\min}$, any distribution $F(\cdot|a)$, and any optimal a^* , it must be that $a^* < \hat{a}$. Since $\hat{a} > 0$ is arbitrary, $\lim_{\underline{y} \rightarrow -\infty} a^* = 0$.

Suppose $\pi(y) \equiv y$. To prove that a^* is increasing in $\underline{y}$, it suffices to show that the principal's payoff from implementing a in an optimal contract, $\Pi(a, \underline{y}) = a - c'(a)(a - \underline{y}) + w$, is supermodular in a and $\underline{y}$.

Recall that $w = \min\{M, c'(a)(a - \underline{y}) - c(a) - u_0\}$ is a function of $(a, \underline{y})$. Therefore,

$$\frac{\partial \Pi}{\partial a} = 1 - c''(a)(a - \underline{y}) - c'(a) + \frac{\partial w}{\partial a}$$

and so

$$\frac{\partial^2 \Pi}{\partial y \partial a} = c''(a) + \frac{\partial^2 w}{\partial y \partial a}.$$

But $\frac{\partial^2 w}{\partial y \partial a} = 0$ if $M < c'(a)(a - \underline{y}) - c(a) - u_0$ and $\frac{\partial^2 w}{\partial y \partial a} = -c''(a)$ otherwise. In either case, $\frac{\partial^2 \Pi}{\partial y \partial a} \geq 0$ and so optimal effort a^* is increasing in $\underline{y}$, as desired. $\square$

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