

Supplement to “Dynamic project selection”
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APPENDIX: AUXILIARY TECHNICAL LEMMAS

LEMMA A.1. (i) When $c < \underline{c}$, $\Phi^C(\cdot, c)$ on $[\underline{x}, \bar{x}]$ is positive at first, then intersects zero at a point, then is negative, then intersects zero at a point, and then is positive again.

(ii) The $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$ if and only if $c \geq \underline{c}$.

(iii) The $\Phi^A(\cdot, \cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]^2$ if and only if $c \geq \underline{c}$.

(iv) When $c < \bar{c}$, $\Phi^A(x_1, \cdot, c)$ is quasi-convex on $[x_1, b^{-1}(x_1)]$.

PROOF. The proof proceeds in steps.

Step 1. *Claim.* We have $1 - c - a > 0$ and $1 - c - a^2 > 0$ for any $a \in [\underline{x}, \bar{x}]$.

PROOF. We have

$$1 - c - a > 1 - c - \bar{x} = \frac{1 - 2c - \sqrt{1 - 4c}}{2} = \frac{4c^2}{2(1 - 2c + \sqrt{1 - 4c})} > 0,$$

which, coupled with $a^2 < a$, also implies that $1 - c - a^2 > 0$. $\triangleleft$

Step 2. *Claim.* Define $\hat{a} = 1 - \sqrt{c}$. Then $\hat{a} \in (\underline{x}, \bar{x})$, and $a \in [\underline{x}, \hat{a}] \cup (\hat{a}, \bar{x}]$ implies that $(a - \hat{a})(c - (1 - a)^2) > 0$.

PROOF. The inequality follows by the definition of $\hat{a}$ and by inspection. It remains to verify that $\hat{a} \in (\underline{x}, \bar{x})$. Indeed,

$$\begin{aligned}\hat{a} - \underline{x} &= \frac{1 + \sqrt{1 - 4c} - 2\sqrt{c}}{2} > 0, \\ \bar{x} - \hat{a} &= \frac{2\sqrt{c} + \sqrt{1 - 4c} - 1}{2} > 0,\end{aligned}$$

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where $c < \bar{c}$. $\triangleleft$

Step 3. Claim. If $\Phi^C(a, c) < 0$ for some $a \in [\underline{x}, \bar{x}]$, then $\Phi^C(\cdot, c)$ is at first positive, then intersects zero at a single point to the left of $\hat{a}$, then is negative, then intersects zero at a single point to the right of $\hat{a}$, and then is again positive.

PROOF. Because $\Phi^C(\underline{x}, c) = 1/(1 - \underline{x}) > 0$ and $\Phi^C(\bar{x}, c) = 1/(1 - \bar{x}) > 0$, then $\Phi(a, c) \leq 0 \implies a \in (\underline{x}, \bar{x})$. Differentiating yields

$$\begin{aligned}\Phi_1^C(a, c) &= \frac{c(1 - c - a^2)}{(1 - a)(1 - c - a)^2} + \ln \frac{c(1 - a)}{a(1 - c - a)} \\ &= \frac{\Phi^C(a, c)}{a} + \frac{(c - (1 - a)^2)(1 - c - a^2)}{a(1 - a)(1 - a - c)^2}.\end{aligned}$$

If $\Phi^C(a, c) = 0$ for some $a \in (\underline{x}, \bar{x})$, then

$$\Phi_1^C(a, c) = \frac{(c - (1 - a)^2)(1 - c - a^2)}{a(1 - a)(1 - c - a)^2}.$$

By Step 1, the sign of $\Phi_1^C(a, c)$ is the sign of $c - (1 - a)^2$, which, by Step 2, switches the sign from negative to positive at $\hat{a} \in (\underline{x}, \bar{x})$. Hence, if a with $\Phi^C(a, c) < 0$ exists, then $\Phi^C(\cdot, c)$ intersects zero twice: once from above and to the left of $\hat{a}$, and once from below and to the right of $\hat{a}$. $\triangleleft$

Step 4. Claim. If $c < \underline{c}$, then $\Phi^C(\hat{a}, c) < 0$; if $c \geq \underline{c}$, then $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$.

PROOF. Note that, at $a \in (\underline{x}, \bar{x})$,

$$\Phi_2^C(a, c) = \frac{a}{c} \left(\frac{1 - a}{1 - c - a} \right)^2 > 0.$$

Furthermore,

$$\Phi^C(\hat{a}, \underline{c}) = 2 - (1 - \sqrt{\underline{c}}) \ln \frac{(1 - \sqrt{\underline{c}})^2}{\underline{c}} = 0,$$

where the first equality is by $\hat{a} = 1 - \sqrt{\underline{c}}$ and the second equality is by (8). Combining the two displays above delivers $\Phi^C(\hat{a}, c) < 0$ for any $c < \underline{c}$ and $\Phi^C(\hat{a}, c) \geq 0$ for any $c \geq \underline{c}$.

For $c < \underline{c}$, $\Phi^C(\hat{a}, c) < 0$ and Step 3 imply part (i).

For $c \geq \underline{c}$, $\Phi^C(\hat{a}, c) \geq 0$ and Step 3 imply part (ii). $\triangleleft$

Step 5. Claim. Subject to $x_2 \geq x_1$, $\Phi^A(\cdot, \cdot, c)$ is minimized at $x_2 = x_1$.

PROOF. The claim follows from

$$\Phi_2^A(x_1, x_2, c) = \frac{(1 - x_1)}{(1 - x_2)^2} \left(\ln \frac{c(1 - x_1)}{x_1(1 - c - x_2)} + \frac{(1 - x_2)(1 - c - x_2^2)}{(1 - c - x_2)^2} \right) > 0,$$

where the inequality follows because $1 - c - x_2^2 > 0$ and $1 - c - x_2 > 0$ by Step 1.

Because $\Phi^A(z, z, c) = \Phi^C(z, c)$, Step 5 implies that $\Phi^A(\cdot, \cdot, c)$ has the same minimized value as $\Phi^C(\cdot, c)$ does. Hence, part (iii) is implied by part (ii). $\triangleleft$

Step 6. *Claim.* Define $\kappa(x_1, x_2, c) \equiv (1 - x_2)(2c - 1 + x_2 + x_2^2 - x_1x_2^2) - c^2$. Then, for some $y^A \in (\underline{x}, \bar{x})$ and for any $x_2 \in [\underline{x}, y^*) \cup (y^*, \bar{x}]$, $(x_2 - y^A)\kappa(x_1, x_2, c) > 0$.

PROOF. First, we show that $\kappa(x_1, \bar{x}, c) > 0$. Indeed,

$$\kappa(x_1, \bar{x}, c) \geq \kappa(\bar{x}, \bar{x}, c) = c - (1 - \sqrt{1 - 4c})\left(\frac{1}{2} - c\right) > 0,$$

where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\bar{x}, \bar{x}, \cdot)$ is zero at $c \in \{0, \bar{c}\}$ and is positive at the only critical point ($c = 2/9$) in $(0, \bar{c})$.

Next, we show that $\kappa(x_1, \underline{x}, c) < 0$. Indeed,

$$\kappa(x_1, \underline{x}, c) \leq \kappa(\underline{x}, \underline{x}, c) = c - (1 + \sqrt{1 - 4c})\left(\frac{1}{2} - c\right) < 0,$$

where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\underline{x}, \underline{x}, \bar{c}) = 0$ and because $\partial\kappa(\underline{x}, \underline{x}, c)/\partial c > 0$.

Finally, $\partial^2\kappa(x_1, x_2, c)/x_2^2 = -6(1 - x_1)x_2 - 2x_1 < 0$. Hence, $\kappa(x_1, \underline{x}, c) < 0$ and $\kappa(x_1, \bar{x}, c) > 0$ imply that, on $(\underline{x}, \bar{x})$, $\kappa(x_1, \cdot, c)$ crosses zero and—by $\partial^2\kappa(x_1, x_2, c)/x_2^2 < 0$ —just once, from below, at some $y^A \in (\underline{x}, \bar{x})$. $\triangleleft$

Step 7. *Claim.* The $\Phi^A(x_1, \cdot, c)$ can be negative on and only on an interval.

PROOF. At any (x_1, x_2, c) with $\Phi^A(x_1, x_2, c) = 0$, by differentiation and substitution,

$$\Phi_2^A(x_1, x_2, x) = \frac{\kappa(x_1, x_2, c)}{(1 - x_2)x_2(1 - c - x_2)^2}.$$

The sign of $\Phi_2^A(x_1, x_2, c)$ is the sign of $\kappa(x_1, x_2, c)$, which, by Step 6, switches from negative to positive at $y^A \in (\underline{x}, \bar{x})$ as x_2 rises; $\Phi^A(x_1, x_2, c)$ is quasi-convex. Part (iv) follows. $\square$

LEMMA A.2. *The function M^C is uniquely maximized on $[\underline{x}, \bar{x}]$ at $\bar{x}$.*

PROOF. Recall from the proof of Lemma 3 that M^C has two local maxima: at $\underline{a}$ and at $\bar{x}$. It remains to verify that $M^C(\bar{x}) > M^C(\underline{a})$.

Then

$$M^C(\bar{x}) = 2c\sigma(\bar{x}) - \frac{1 - V(\bar{x}, \bar{x})}{(1 - \bar{x})^2} = 2c\sigma(\bar{x}) - \frac{1}{1 - \bar{x}},$$

where the last equality uses $V(\bar{x}, \bar{x}) = \bar{x}$, by direct substitution. Furthermore,

$$M^C(\underline{a}) = 2c\sigma(\underline{a}) - \frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} = 2c\sigma(\underline{a}) - \frac{c}{\underline{a}(1 - \underline{a})^2} - \frac{1 - \underline{a}}{1 - \underline{a} - c},$$

where the last equality follows by substituting $\Phi(\underline{a}, c) = 0$ into the expression for $\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2}$.

As a result,

$$\begin{aligned} M^C(\bar{x}) - M^C(\underline{a}) &= 2c\sigma(\bar{x}) - 2c\sigma(\underline{a}) + \frac{c}{\underline{a}(1 - \underline{a})^2} + \frac{1 - \underline{a}}{1 - \underline{a} - c} - \frac{1}{1 - \bar{x}} \\ &= c \left[2\sigma(\bar{x}) - \frac{1}{\bar{x}(1 - \bar{x})^2} \right] - c \left[2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right] + \frac{1 - \underline{a}}{1 - \underline{a} - c} > 0, \end{aligned}$$

where the last equality follows from $\bar{x}(1 - \bar{x}) = c$ and by rearranging, and the inequality follows because the first bracket exceeds the second bracket, and the fraction $(1 - \underline{a})/(1 - \underline{a} - c)$ is positive (by $\underline{a} < \bar{x}$). The ordering of the brackets follows from $\bar{x} > \underline{a}$ and the observation

$$\frac{d}{da} \left(2\sigma(a) - \frac{1}{a(1 - a)} \right) = \frac{1}{a^2(1 - a)^2} > 0, \quad \forall a \in (0, 1).$$

To summarize, $M^C(\bar{x}) > M^C(\underline{a})$ and, so, M^C has a unique maximand, $\bar{x}$, on $[\underline{x}, \bar{x}]$. $\square$

LEMMA A.3. For M^A defined in (18), $\arg \max_{a \in [x_1, b^{-1}(x_1)]} M^A(x_1, a) = \{b^{-1}(x_1)\}$, where b^{-1} is the inverse of b defined in (11). As a result, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$.

PROOF. By Lemma 4, the only two local maxima of $M^A(x_1, \cdot)$ are $d(x_1)$ and $b^{-1}(x_1)$, so it suffices to show that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$. Write

$$M^A(x_1, b^{-1}(x_1)) = c\eta(b^{-1}(x_1)) - \frac{1 - V(x_1, b^{-1}(x_1))}{1 - b^{-1}(x_1)} = c\eta(b^{-1}(x_1)) - 1,$$

where the first equality is definitional and the second equality is by $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$.

Evaluating V in (10) at $(x_1, d(x_1))$ and using $\Phi^A(x_1, d(x_1), c) = 0$ (by (19)), one can write

$$V(x_1, d(x_1)) = (1 - x_1)d(x_1) - \frac{(1 - d(x_1))c(1 - c - x_1d(x_1))}{d(x_1)(1 - c - d(x_1))} - c + x_1.$$

Then

$$\begin{aligned} M^A(x_1, d(x_1)) &= c\eta(d(x_1)) - \frac{1 - V(x_1, d(x_1))}{1 - d(x_1)} \\ &= c \left[\eta(d(x_1)) - \frac{1}{d(x_1)(1 - d(x_1))} \right] + x_1 - \frac{c(1 - x_1)}{1 - c - d(x_1)} - 1, \end{aligned}$$

where the first equality is definitional and the second equality follows by substituting $V(x_1, d(x_1))$ and rearranging.

Then, suppressing the argument x_1 in $b^{-1}(x_1)$ and in $d(x_1)$, for compactness,

$$\begin{aligned} M^A(x_1, b^{-1}) - M^A(x_1, d) &= c \left[\eta(b^{-1}) - \frac{1}{b^{-1}(1-b^{-1})} - \left(\eta(d) - \frac{1}{d(1-d)} \right) \right] \\ &\quad + \frac{b(d)(b^{-1} - x_1) + x_1(1-b^{-1})}{b^{-1}(1-b(d))} > 0, \end{aligned}$$

where the first equality follows by using the definitions of b and b^{-1} and rearranging, and the inequality uses $b^{-1}(x_1) > d(x_1)$ and

$$\frac{d}{dy} \left(\eta(y) - \frac{1}{y(1-y)} \right) = \frac{1}{y^2(1-y)} > 0$$

to conclude that the bracket in the first line is positive; and uses $x_1 < b^{-1}(x_1) < 1$ to conclude that the fraction in the second line is positive too. That is, $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$, as desired.

The conclusion that, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$ follows by inspection of Lemma's 4 Figure 7 (just validated by showing that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$). $\square$

LEMMA A.4. On $\hat{\mathcal{B}}$, $w(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{B}$.

PROOF. To conclude that $w(x_1) < b^{-1}(x_1)$, we show that $V(x_1, b^{-1}(x_1)) > B(x_1, b^{-1}(x_1))$.

Note that $V(x_1, b^{-1}(x_1)) = b^{-1}(x_1)$ and, from the definition of B in (22),

$$B(x_1, b^{-1}(x_1)) = 1 - (1 - b^{-1}(x_1)) \left(\frac{1 - C(x_1)}{1 - x_1} + c\eta(b^{-1}(x_1)) - c\eta(x_1) \right).$$

Then

$$V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) = (1 - b^{-1}(x_1)) \left(\frac{1 - C(x_1)}{1 - x_1} - 1 + c\eta(b^{-1}(x_1)) - c\eta(x_1) \right),$$

where

$$\begin{aligned} \frac{1 - C(x_1)}{1 - x_1} &= (1 - x_1) \left(\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right) \\ &= (1 - x_1) \left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right). \end{aligned}$$

The first equality in the display above uses the definition of C in (13). The second equality uses the definitions of V in (10) and b in (11), and the condition $\Phi^C(\underline{a}, c) = 0$ in (15), which characterizes $\underline{a}$.

Then, substituting the display above into its precursor display gives

$$\begin{aligned}
& \frac{V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1))}{1 - b^{-1}(x_1)} \\
&= (1 - x_1) \left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c[\sigma(x_1) - \sigma(\underline{a})] \right) \\
&\quad + c[\eta(b^{-1}(x_1)) - \eta(x_1)] - 1 \\
&= \left(\frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - 1 \right. \\
&\quad \left. + (1 - x_1)c \left[2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right) \right] \right. \\
&\quad \left. + c \left[\eta(b^{-1}(x_1)) - \frac{1}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)} \right) \right] \right).
\end{aligned}$$

Note that, using the definition of b in (11),

$$\begin{aligned}
& \frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - 1 \\
&= \frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{x_1}{b^{-1}(x_1)} - 1 \\
&= \frac{x_1[1 - b^{-1}(x_1)] + b(\underline{a})[b^{-1}(x_1) - x_1]}{b^{-1}(x_1)(1 - b(\underline{a}))} > 0,
\end{aligned}$$

where the inequality follows from $x_1 < b^{-1}(x_1) < 1$. Moreover,

$$2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2} \right) > 0$$

by $x_1 > \underline{a}$ and by

$$\frac{d}{dy} \left(2\sigma(y) - \frac{1}{y(1 - y)^2} \right) = \frac{1}{y^2(1 - y)^2} > 0$$

for any $y \in (0, 1)$. Finally,

$$\eta(b^{-1}(x_1)) - \frac{1}{b^{-1}(x_1)(1 - b^{-1}(x_1))} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)} \right) > 0$$

by $b^{-1}(x_1) > x_1$ and by

$$\frac{d}{dy} \left(\eta(y) - \frac{1}{y(1 - y)} \right) = \frac{1}{y^2(1 - y)} > 0$$

for $y \in (0, 1)$. Thus, $V(x_1, b^{-1}(x_1)) - B(x_1, b^{-1}(x_1)) > 0$, as required.

To show that $(\mathcal{F} \cap \hat{\mathcal{B}}) \subset \mathcal{B}$, from (22) and (10), write

$$\begin{aligned} \frac{B(x) - V(x)}{1 - x_2} &= (1 - x_1) \left(\frac{1}{1 - b(x_2)} + \frac{c}{1 - x_2} [\eta(x_1) - \eta(b(x_2))] \right) \\ &\quad - \frac{1 - C(x_1)}{1 - x_1} - c[\eta(x_2) - \eta(x_1)]. \end{aligned}$$

Differentiating and then simplifying gives

$$\frac{d}{dx_2} \left(\frac{B(x) - V(x)}{1 - x_2} \right) = - \frac{c\Phi^A(x, c)}{x_2(1 - x_2)}.$$

As a result, because $\Phi^A(x, c) < 0$ implies that $B(x) > V(x)$, $\mathcal{B}$ covers $\mathcal{F}$, the failure region, on $\hat{\mathcal{B}}$. That is, $\mathcal{F} \cap \hat{\mathcal{B}} \subset \mathcal{B}$. $\square$

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