Theoretical Economics 16 (2021), 881–909 1555-7561/20210881

Comparing school choice and college admissions mechanisms by their strategic accessibility

Somouaoga Bonkoungou
Faculty of Business and Economics, University of Lausanne

Alexander Nesterov
Department of Economics and Game Theory Lab, Higher School of Economics

Dozens of school districts and college admissions systems around the world have reformed their admissions rules in recent years. As the main motivation for these reforms, the policymakers cited the strategic flaws of the rules in place: students had incentives to game the system. However, after the reforms, almost none of the new rules became strategy-proof. We explain this puzzle. We show that the rules used after the reforms are less prone to gaming according to a criterion called "strategic accessibility": each reform expands the set of schools wherein each student can never get admission by manipulation. We also show that the existing explanation of the puzzle due to Pathak and Sönmez (2013) is incomplete.

Keywords. Market design, school choice, manipulability.

JEL classification. C78, D47, D78.

1. Introduction

In recent years, many school districts around the world have reformed their school admissions systems. Examples include education policy reforms for the K-9 Boston Public Schools (PS) in 2005, the Chicago Selective High Schools (SHS) in 2009 and 2010, the Denver Public Schools in 2012, the Seattle Public Schools in 1999, the Ghanaian Secondary Public Schools in 2007, and the English Public Schools between 2005 and 2011. The series of reforms concern college admissions as well, such as college admissions in China and Taiwan.

Somouaoga Bonkoungou: bkgsom@gmail.com
Alexander Nesterov: nesterovu@gmail.com

We thank Parag Pathak and Tayfun Sönmez for their comments and continuous communication. We also thank Anna Bogomolnaia, Yan Chen, Battal Doğan, Lars Ehlers, Rustamdjan Hakimov, Sean Horan, Onur Kesten, Fuhito Kojima, Herve Moulin, Mikhail Panov, Szilvia Papai, Alvin Roth, and Fedor Sandomirskiy for their valuable feedback. We are especially thankful to Lars Ehlers, Sean Horan, and the anonymous referee for their help in revising the manuscript. This paper supersedes an older version titled as "Comparing School Choice and College Admissions Mechanisms by Their Immunity to Strategic Admissions." Support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. The research is partially financed by the Russian Foundation for Basic Research grants 19-01-00762, 20-01-00687. Somouaoga's work in this project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 890648. The results of this publication reflect only the authors' view and the commission is not responsible for any use that may be made to the information it contains.

© 2021 The Authors. Licensed under the Creative Commons Attribution-NonCommercial License 4.0. Available at https://econtheory.org. https://doi.org/10.3982/TE4137

Sometimes the reforms were a pressing issue. The Chicago SHS, for example, called for reform midstream in their admissions process. What were the policymakers concerned about, and what was at stake for such a sudden midstream change? It was widely reported that the concern behind all these reforms was an excessive vulnerability of the admissions mechanisms to manipulations. For example, the former superintendent of the Boston PS said that their mechanism *should be replaced with an alternative [...] that removes the incentives to game the system* (Pathak and Sönmez 2008).

Indeed, manipulability made strategy an essential decision for students and led to serious mismatches. Strategic sophistication was playing an unbalanced role in the admissions versus priorities/grades, and that was perceived as undesirable. For example, the Chicago SHS called for reform after they observed that *High-scoring kids were being rejected simply because of the order in which they listed their college prep preferences* (Pathak and Sönmez 2013). Prior to the reform, one parent in China remarked that, *a good score in the college entrance exam is worth less than a good strategy in the ranking of colleges* (Chen and Kesten 2017, Nie 2007).

Did the reforms remove all possibilities for manipulation? The answer is no. Every reform, except Boston PS, replaced one vulnerable mechanism with another vulnerable mechanism. Having said that, one would expect the new mechanisms to be *less* vulnerable to manipulation than the ones in place prior to the reforms. In this paper, we develop a criterion to investigate this conjecture.

To explain our criterion, let us begin with the robust mechanisms. A mechanism is strategy-proof when no student can benefit by misrepresenting her preferences. In such a mechanism, no school is accessible to any student by manipulation. Let us generalize this definition to any, possibly nonstrategy-proof, mechanism. We say that school s is *not strategically accessible* to student i via mechanism φ if for any problem there is no profitable misreport by which i could be placed at s via φ. In other words, all the misreports that result in i's admission to school s via mechanism φ are not profitable. Student i may still profitably manipulate the mechanism and be strategically admitted at other schools—but not at s.

We measure the level of vulnerability of each mechanism by the set of schools, which are not strategically accessible to each student. We deem mechanism φ *less strategically accessible than* ψ if, for each student, every school which is not strategically accessible to her via ψ is also not strategically accessible to her via φ, while there is a school choice context where the converse is not true.

The main result of this paper is that the mechanism used after each of the reforms mentioned above is less strategically accessible than the mechanism used before the reform (Theorems 1, 2, 3, Table 2). Simultaneously, following the reforms, each school became strategically accessible to a weakly smaller set of students.

Roughly, the reforms resulted in mechanisms that were less strategically accessible by mandating one or both of the following. First, they allowed students to submit longer lists of acceptable schools. Second, for the submitted lists, they switched from the immediate acceptance to the deferred acceptance procedure. Intuitively, a longer list allows students to be less strategic about selecting which schools to include in the list, while deferred acceptance facilitates the truthful ranking of the selected schools. We illustrate the concept and the result in the following example.

#	School name	US rank	min score	max score
1	Payton	9	898	900
2	Northside	23	894	900
3	Lane	69	873	900
4	Young	71	883	900
5	Jones	91	891	900
6	Brooks	186	799	890
7	Lindblom	272	772	858
8	Westinghouse	574	773	884
9	King	1133	678	844
10	South Shore	6066	684	820

Note: South Shore was added to SHS in 2011, followed by Hancock school in 2015. The US rank is according to US News & World 2019 Report; the grades are reported for general admission in 2019, the maximal feasible grade is 900; the data source is www.go.cps.edu.

TABLE 1. Chicago selective high schools (SHS): rankings and cutoff grades.

Illustrative example

We consider two reforms of the Chicago SHS, in 2009 and 2010. Each school uses a common priority based on students' composite scores. The admission to each of these schools is very competitive. To give you an idea, in the 2018 admissions session, only 4000 out of more than 10,000 participants were admitted. In 2009, the Chicago SHS replaced the immediate acceptance mechanism (a.k.a. Boston mechanism) where students can rank only four schools (β^4) with a serial dictatorship with the same constraint (SD^4).[1] In 2010, they kept the serial dictatorship procedure but extended the ranking constraint to six (SD^6).

Five of the ten schools are elite schools. They are the top five schools in the state of Illinois and are among the top 100 in the US (see Table 1). These schools are preferred by most, if not all, students over the other schools. For simplicity, let us suppose that students have tier preferences. That is, each student prefers each elite school over each of the nonelite schools, but students may rank schools in each tier differently. Let each school have 400 seats.

Under the mechanism β^4, each of the 400-highest priority students is guaranteed a seat at her most preferred school, while every other student may need to manipulate the mechanism to gain admission to any given school. Thus, under β^4 each school is not strategically accessible only to the 400-highest priority students.

However, under the mechanism SD^4, each of the 1600-highest priority students is guaranteed one of her four most preferred schools. No school is strategically accessible to any of them (Proposition 2), while every school is strategically accessible to each of the other students. Under SD^6, this set represents the 2400-highest priority students; while each elite school is no longer strategically accessible to any student. Thus, following the reforms, strategic accessibility to schools decreased and the share of students for whom

[1]The serial dictatorship is a mechanism where students follow the common priority order and choose their most preferred schools among those that remain. The definitions of the mechanisms are given in the next section.

the elite schools were not strategically accessible increased from 4% to 24% in 2009 and further to 100% in 2010.

The state-of-the-art for explaining these reforms is the notion of *manipulability* proposed by Pathak and Sönmez (2013). Mechanism φ is less manipulable than ψ if at every preference profile where φ is manipulable by at least some student, ψ is also manipulable by at least some student, and there is a school choice context where the reverse does not hold.

We show that this notion does not apply to some reforms. First, we show that manipulability only partially explains the reforms in England. These reforms followed a single act of Parliament banning the so-called first-preference-first admission criterion. If all schools in the system used this principle before the reform, then the mechanism is equivalent to the Boston mechanism, and the reform resulted in a less manipulable mechanism. However, if only some schools used this principle, the reform did not result in a less manipulable mechanism (Table 2). We provide a counterexample to a corresponding result by Pathak and Sönmez (2013).

Second, we argue that when preferences are homogeneous, manipulability does not distinguish the constrained serial dictatorship and the constrained Boston mechanism. Indeed, it suffices that one student has a profitable manipulation to declare a mechanism as manipulable at a preference profile. However, with tier preferences as in the example above—competition for the elite schools—at least one such student always exists. More generally, we show that the constrained serial dictatorship mechanism is not manipulable if and only if the constraint is not binding for every student (Proposition 2). In contrast, strategic accessibility to schools changed significantly after both the 2009 and 2010 reform, and the fact that elite schools are no longer strategically accessible may explain why SD^6 has been used in Chicago ever since.

Strategic accessibility was first formulated by Bonkoungou (2018) to rank a class of mechanisms that favor the higher ranking of schools. Recently, Arribillaga and Massó (2016) ranked voting rules by the set inclusion of the vulnerable preference relations of each agent, that is, the relations for which there exist preferences of others such that this agent can manipulate. This notion was recently used by Decerf and Van der Linden (2020) to rank the constrained Boston mechanism and the constrained deferred acceptance mechanism as well as different constraints of the deferred acceptance mechanism. Andersson et al. (2014a) ranked budget balanced and fair rules by counting, for each preference profile, the number of agents with the incentive to manipulate. They find rules that minimize the number of agents and coalitions that can manipulate. In the same problem, Andersson et al. (2014b) find the rule that minimizes the maximal gain that an agent can get by manipulation. Next, Chen and Kesten (2017) and Dur et al. (forthcoming) used manipulability to compare mechanisms in China and Taiwan, respectively. Chen et al. (2016) formulated another notion by counting and comparing, for each preference profile, the set of outcomes that each agent can get via manipulation. This notion is useful for ranking stable matching mechanisms. The main difference with our notion is that it is a preference by preference comparison, and cannot explain any of the reforms studied here (except the Chicago SHS in 2010).

In two companion papers, we give a complementary rationale for a subset of reforms in school admissions systems. The reforms also decreased the set of instances where the outcome is unstable, and some reforms decreased the number of students with justified envy, Bonkoungou and Nesterov (2020b), and the number of students with an incentive to manipulate, Bonkoungou and Nesterov (2020a), which strengthens the results in Pathak and Sönmez (2013).

The paper is organized as follows. In Section 2, we present the model and the main definitions. In Section 3, we present the main results. In Section 4, we compare them to the results for manipulability. In Section 5, we develop an equilibrium refinement of strategic accessibility. In Section 6, we conclude. We collect some proofs in the Appendix.

2. Model

There is a finite and nonempty set I of students with a generic element i and a finite and nonempty set S of schools with a generic element s. Each student i has a strict preference relation P_i over $S \cup \{\varnothing\}$ (where $\varnothing$ represents the outside option for this student). Each school s has a strict priority order $\succ_s$ over I and a capacity q_s (a natural number representing the maximal number of students that this school can admit). For each student i, let R_i denote the "at least as good as" relation associated with P_i.[2] School s is *acceptable* to student i if $s\ P_i\ \varnothing$, and unacceptable if $\varnothing\ P_i\ s$. The list $P = (P_i)_{i\in I}$ is a preference profile, $\succ = (\succ_s)_{s\in S}$ is a priority profile and $q = (q_s)_{s\in S}$ is a capacity vector. We often write a preference profile $P = (P_i, P_{-i})$ to emphasize the preference relation of student i.[3]

We extend the priority order $\succ_s$ of each school s over I to the set 2^I of subsets of students and assume that it is responsive to the priority order over I (Roth 1985). By definition, the priority order $\succ_s$ over 2^I is responsive if for any students $i, j \in I$ and any subset $N \subset I \setminus \{i, j\}$ such that $|N| < q_s$, then (i) $N \cup \{i\} \succ_s N$, and (ii) $N \cup \{i\} \succ_s N \cup \{j\}$ if and only if $i \succ_s j$.

The tuple $(I, S, \succ, q)$ is a school choice *context* and $(I, S, P, \succ, q)$ a school choice *problem*. In order to reflect school choice in real life, we assume that there are at least two schools and more students than schools, $|I| > |S| \geq 2$. We assume that the school choice context $(I, S, \succ, q)$ is fixed, such that the preference profile determines the problem.[4]

A matching μ is a function $\mu : I \to S \cup \{\varnothing\}$ such that for each school s, $|\mu^{-1}(s)| \leq q_s$. We say that student i is unmatched when $\mu(i) = \varnothing$.

Next, we define some basic terms. Let P be a preference profile. A matching μ is individually rational under P if for each student i, $\mu(i)\ R_i\ \varnothing$. Student i has justified envy toward student j under the matching μ if there is a school s such that $s\ P_i\ \mu(i)$, $\mu(j) = s$

[2] That is, for each $s, s' \in S \cup \{\varnothing\}$, $s\ R_i\ s'$ if and only $s\ P_i\ s'$ or $s = s'$.

[3] More generally, we write a preference profile $P = (P_{I'}, P_{-I'})$ to emphasize the components of a subset I' of students.

[4] We assume that schools are not strategic. In practice, the priorities are determined by law or by students' performances, and are known to students before they submit their preferences.

and $i \succ_s j$. A matching μ is nonwasteful if for each student i, there is no school s such that $s\, P_i\, \mu(i)$ and $|\mu^{-1}(s)| < q_s$. A matching is *stable* under P if it is individually rational, no student has a justified envy toward another, and it is nonwasteful.

A mechanism φ is a function that maps each problem P to a matching. Let $\varphi_i(P)$ denote the assignment for student i.

2.1 *Mechanisms*

Gale–Shapley In a seminal paper, Gale and Lloyd (1962) showed that for each problem, there exists a stable matching. They also showed that there is a matching, the student-optimal stable matching, where each student finds her assignment at least as good as her assignment at any other stable matching. They proposed the following student-proposing deferred acceptance algorithm to find this matching.

- *Step 1:* Each student applies to her most preferred acceptable school (if any). If a student did not rank any school acceptable, then she is unmatched. Let I_s^1 denote the applicants of school s at this step. Each school s tentatively accepts $\min(q_s, |I_s^1|)$ of the $\succ_s$-highest priority applicants and rejects the remaining ones. Let A_s^1 denote the tentative acceptances of school s at this step.

- *Step t, $t > 1$:* Each student who is rejected at step $t-1$ applies to her most preferred acceptable school among those she has not yet applied to (if any). If a student is rejected by all of her acceptable schools, then she is unmatched. Let I_s^t denote the new applicants of school s at this step. Each school s tentatively accepts $\min(q_s, |A_s^{t-1} \cup I_s^t|)$ of the $\succ_s$-highest priority applicants and rejects the remaining ones. Let A_s^t denote the tentative acceptances of school s at this step.

The algorithm stops when every student is either tentatively accepted or rejected by all of her acceptable schools. The tentative acceptances at this step become the final matching. The Gale–Shapley mechanism GS assigns to each preference profile P the matching $\mathrm{GS}(P)$ obtained by this algorithm.

Serial dictatorship In every school choice context $(I, S, \succ, q)$ where schools have the same priority order, that is, for each $s, s' \in S$, $\succ_s = \succ_{s'}$, the Gale–Shapley mechanism is called a serial dictatorship mechanism. Let $\mathrm{SD}(P)$ denote the matching assigned by this mechanism to each preference profile P. The mechanism can be described in the following simple procedure. The student who is ordered first by the priority order picks her most preferred acceptable school (if any). The student ordered next picks her most preferred acceptable school among those remaining (if any), and so on.

Boston Abdulkadiroğlu and Sönmez (2003) describe the following immediate acceptance mechanism that Boston PS was using until 2005:

- *Step 1:* Each student applies to her most preferred acceptable school (if any). If a student did not rank any school acceptable, then she is unmatched. Let I_s^1 denote the applicants of school s at this step. Each school s immediately accepts $\min(q_s, |I_s^1|)$ of the $\succ_s$-highest priority applicants and rejects the remaining ones.

For each school s, let $q_s^1 = q_s - \min(q_s, |I_s^1|)$ denote its remaining capacity after this step.

- *Step t, $t > 1$*: Each student who is rejected at step $t-1$ applies to her most preferred acceptable school among those she has not yet applied to (if any). If a student is rejected by all of her acceptable schools, then she is unmatched. Let I_s^t denote the applicants of school s at this step. Each school s immediately accepts $\min(q_s^{t-1}, |I_s^t|)$ of the $\succ_s$-highest priority applicants and rejects the remaining ones. Let $q_s^t = q^{t-1} - \min(q_s^{t-1}, |I_s^t|)$ denote the remaining capacity of school s after this step.

The algorithm stops when each student is either immediately accepted or rejected by all her acceptable schools. Every school is assigned to the students that it accepted at each step. The Boston mechanism assigns to each preference profile P, the matching $\beta(P)$ obtained by this algorithm. The Boston mechanism is individually rational but does not always produce a stable matching (Abdulkadiroğlu and Sönmez 2003).

First-preference-first Pathak and Sönmez (2013) describe a mechanism that the school boards of many English cities were using until 2007 when the Parliament banned its use throughout the country. In this system, each school is either an *equal preference school* or *first-preference-first school*. This distribution is exogenous and is a parameter for the mechanism. In the mechanism, the original priority of each equal preference school remains unchanged, while the original priority of each first-preference-first school is adjusted according to the ranking of schools in the following way.

Let P be a preference profile. For each school s, let I_s^1 denote the set of students who ranked s first (including the ranking of the outside option $\varnothing$), I_s^2 the set of students who ranked it second, I^3 the set of students who ranked it third, and so on. Let $\hat{\succ}_s$ denote the following adjusted priority order:

1. for each equal preference school s, $\hat{\succ}_s = \succ_s$ and
2. for each first-preference-first school s,
   - for each $i \in I_s^1$ and each $j \in I \setminus I_s^1$, we have $i \,\hat{\succ}_s\, j$. For each $\ell \in \{2, \ldots, |S|\}$, each $i \in I_s^\ell$ and each $j \in I \setminus (I_s^1 \cup \cdots \cup I_s^\ell)$, we have $i \,\hat{\succ}_s\, j$.
   - for each $\ell \in \{1, \ldots, |S|+1\}$ and each $i, j \in I_s^\ell$ such that $i \succ_s j$, we have $i \,\hat{\succ}_s\, j$.

The First-Preference-First (FPF) mechanism assigns to each preference profile P, the student-optimal stable matching under P where the priority order has been replaced by $\hat{\succ}$. Let $\mathrm{FPF}(P)$ denote this matching.

Remark. In the immediate acceptance algorithm, at each step, students applying to the same school have assigned to it the same rank. Therefore, students applying to a school at a given step of the algorithm rank this school higher than those applying to it at any step after. In particular, no student could be rejected by a school while another student, who has assigned a lower rank to it, is accepted. Thus, the Boston mechanism is a First-Preference-First mechanism where every school is a first-preference-first school. This result follows from Proposition 2 of Pathak and Sönmez (2008).

Note also that the Gale–Shapley mechanism is a First-Preference-First mechanism when each school is an equal preference school.

Constrained versions In practice, school districts restrict the number of schools that each student is allowed to list. This practice is first studied by Haeringer and Klijn (2009). Let $k \in \{1, \dots, |S|\}$. For each student i, the truncation after the k'th acceptable school (if any) of the preference relation P_i with x acceptable schools is the preference relation P_i^k with $\min(x, k)$ acceptable schools such that all schools are ordered as in P_i. The constrained version φ^k of the mechanism φ assigns to each problem P the matching $\varphi^k(P) = \varphi(P^k)$.

Chinese parallel Chen and Kesten (2017) describe the parametric mechanisms that the school boards of many Chinese provinces have been using. Let $e \geq 1$ be a natural number and P a preference profile. The procedure has multiple rounds. In the first round, students are matched according to the constrained Gale–Shapley mechanism GS^e. The matching is final for matched students, while unmatched students proceed to the next round. We reduce the capacity of each school by the number of the students matched to it in this round and we remove the students matched in this round. For each unmatched student, we remove her top-e schools that rejected her in this round. In the next round, the remaining students are again matched according to the constrained Gale–Shapley mechanism GS^e with the remaining capacities and the new preference profile. The matched students are removed, the capacities are decreased, and the profile is updated the same way as before. The procedure continues until either every student is matched or no seat remains. The Chinese mechanism $\text{Ch}^{(e)}$ with parameter e assigns to each preference profile P the matching $\text{Ch}^{(e)}(P)$ found at the end of this procedure.[5]

3. Results

We motivate our definition from the following question. When can we say that the admission of student i to school s via mechanism φ is not due to manipulation? Obviously, if there is no profitable misreport by which i could gain admission at s via mechanism φ, then any admission of student i to school s via φ cannot be attributed to manipulation. When this is true for all students and all schools, then the mechanism is strategy-proof. Formally, φ is *strategy-proof* if for each student i, there is no preference profile P and a preference relation $\hat{P}_i$ such that

$$\varphi_i(\hat{P}_i, P_{-i}) \; P_i \; \varphi_i(P).$$

Under a strategy-proof mechanism, no admission is due to manipulation. However, even in nonstrategy-proof mechanisms some admissions are not due to manipulation.

[5]This definition of the Chinese parallel mechanisms is given only for a class (of the symmetric version) where each round has the same length e. See Chen and Kesten (2017) for details.

DEFINITION 1. Let φ be a mechanism. School s is *not strategically accessible* to student i via φ if there is no preference profile P and a preference relation $\hat{P}_i$ such that

$$s = \varphi_i(\hat{P}_i, P_{-i})\, P_i\, \varphi_i(P).$$

Otherwise, school s is strategically accessible to student i via φ. Mechanism φ is *strategy-proof for a student* if no school is strategically accessible to her via φ.

The unconstrained Gale–Shapley GS mechanism, for example, is strategy-proof (Roth 1982, Dubins and Freedman 1981).

DEFINITION 2. Mechanism φ is *less strategically accessible* than ψ if (i) every school which is not strategically accessible to a student via ψ is also not strategically accessible to her via φ, and (ii) there is a school choice context where a school is not strategically accessible to a student via φ but via ψ.

3.1 *Reforms and strategic accessibility*

In this section, we present our main results: the comparisons of the mechanisms before and after the reforms. All proofs are in the Appendix.

In England and Wales According to the field observation (see Table 2), more than 50 cities in England and Wales have replaced constrained FPF mechanisms with constrained GS mechanisms. In the following theorem, we show that the replacement mechanisms are less strategically accessible.

THEOREM 1. *Let $k > 1$ and suppose that there are at least k schools and at least one first-preference-first school. The constrained Gale–Shapley mechanism* GS^k *is less strategically accessible than the constrained First-Preference-First mechanism* FPF^k.

The school boards of Newcastle have replaced a constrained FPF with a constrained GS but with a longer list. This replacement also resulted in a less strategically accessible mechanism:

COROLLARY 1. *Let $k > \ell$ and suppose that there are at least k schools and at least one first-preference-first school. The constrained Gale–Shapley mechanism* GS^k *is less strategically accessible than the constrained First-Preference-First mechanism* FPF^ℓ.

In Chicago and Denver In 2009, the Chicago Selective High Schools moved from a constrained Boston to a constrained serial dictatorship. A similar replacement has been observed in Denver and four other cities in England. Since the Boston mechanism is a special case of the First-Preference-First mechanism FPF, and the serial dictatorship mechanism SD a special case of the Gale–Shapley mechanism, we make the following conclusion.

COROLLARY 2. *Let $k \geq \ell$ and suppose that there are at least k schools. The constrained Gale–Shapley mechanism* GS^k *is less strategically accessible than the constrained Boston mechanism* β^ℓ.

Allocation system	Year	From	To	Manipulable? (More or less?)	Strategically accessible? (More or less?)
Boston PS	2005	Boston	GS	Less	Less
Chicago SHS	2009	Boston^4	SD^4	Less	Less
	2010	SD^4	SD^6	Less	Less
Ghana–Sec. Schools	2007	GS^3	GS^4	Less	Less
	2008	GS^4	GS^6	Less	Less
Denver Public Schools	2012	Boston^2	GS^5	Less	Less
Seattle Public Schools	1999	Boston	GS	Less	Less
	2009	GS	Boston	More	More
England and Wales					
48 cities	2006/07	FPF^3	GS^3	Not comparable	Less
1 city	2010	GS^3	GS^4	Less	Less
1 city	2010	GS^3	GS^6	Less	Less
4 cities	2005/06	FPF^6	GS^6	Not comparable	Less
1 city	2007	FPF^7	GS^7	Not comparable	Less
1 city	2005	FPF^4	GS^4	Not comparable	Less
1 city	2007	Boston^3	GS^4	Less	Less
1 city	2007	FPF^3	GS^4	Not comparable	Less
1 city	2005	Boston^3	GS^3	Less	Less
3 cities	2007	Boston^3	GS^3	Less	Less
China					
Jiangsu	2001-12	Ch$^{(1)}$	Ch$^{(3)}$	Less	Less
5 provinces	2001-12	Ch$^{(1)}$	Ch$^{(4)}$	Less	Less
5 provinces	2001-12	Ch$^{(1)}$	Ch$^{(5)}$	Less	Less
Hainan	2001-12	Ch$^{(1)}$	Ch$^{(6)}$	Less	Less
Tibet	2001-12	Ch$^{(1)}$	Ch$^{(10)}$	Less	Less

TABLE 2. School and college admissions reforms.

In Chicago and Ghana In 2010, the Chicago SHS again replaced its constrained serial dictatorship with a version with a longer list. In 2007, the Ghanaian Secondary Schools undertook a similar change, from a constrained GS to a version with a longer list and extended the list again in 2008. These types of changes have also been observed in Newcastle (2010) and Surrey (2010) in England.

THEOREM 2. *Let $k > \ell$ and suppose that there are at least k schools. The constrained Gale–Shapley mechanism* GS^k *is less strategically accessible than the constrained Gale–Shapley mechanism* GS$^\ell$.

In Table 2, we list all reforms in school choice and college admissions. The table is constructed using information from Pathak and Sönmez (2013) and Chen and Kesten (2017). We also feature those reforms that are comparable á la Pathak and Sönmez (2013) and those that are not (see Definition 3 and Section 4 for the results).

In college admissions in China Starting from 2001, the school boards of many Chinese provinces changed their mechanisms from the Boston mechanism to various other parallel mechanisms.

THEOREM 3. *Let $e > e'$ and suppose that there are at least e schools. The Chinese mechanism* $\mathrm{Ch}^{(e)}$ *is less strategically accessible than the Chinese mechanism* $\mathrm{Ch}^{(e')}$.

The proofs of the results above are constructive and use similar arguments. For each school choice context, we assume that some school s is strategically accessible to some student i via a new mechanism. The fact that s is strategically accessible to i has certain implications about the priority profile $\succ$. Knowing this, we construct a preference profile and show that s is strategically accessible to i via the old mechanism.

4. COMPARISON WITH MANIPULABILITY

We introduce the notion of manipulability due to Pathak and Sönmez (2013), and compare it with our notion. Given a mechanism φ, a preference profile P is vulnerable under φ if there exists a student i and a preference relation $\hat{P}_i$ such that

$$\varphi_i(\hat{P}_i, P_{-i})\ P_i\ \varphi_i(P).$$

DEFINITION 3 (Pathak and Sönmez 2013). Mechanism φ is *less manipulable* than ψ if (i) every preference profile which is vulnerable under φ is also vulnerable under ψ and (ii) there is a preference profile not vulnerable under φ but vulnerable under ψ.

Broadly, whenever a student has a profitable manipulation at a preference profile P under φ, then at least one student (possibly more) has a profitable manipulation at the same profile P under ψ, while the reverse is not true in some school choice context. It is important to note that a manipulation of one student is enough to declare a preference profile as vulnerable under a mechanism. Comparing mechanisms with respect to a certain property profile by profile is common in the literature. A notable example is Kesten (2006).

4.1 *Limitation in England and Wales*

Recall that the school boards of many English cities have replaced the constrained First-Preference-First mechanism with a constrained Gale–Shapley mechanism. Pathak and Sönmez (2013) claim that this resulted in a less manipulable mechanism.

CLAIM 1 (Pathak and Sönmez 2013, Proposition 3). *Suppose that there are at least k schools where $k > 1$. Then the constrained Gale–Shapley mechanism* GS^k *is less manipulable than the constrained First-Preference-First mechanism* FPF^k.

We provide a counterexample to this claim. We specify the relevant part of the priorities such that the incomplete part indicates that this part is irrelevant and omitted.

EXAMPLE 1 (Counterexample to Claim 1). We consider a problem with seven students and seven schools. Each school has one seat. We specify a preference profile and a priority profile below.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$	$\succ_{s_7}$	$\hat{\succ}_{s_5}$
s_1	s_1	s_2	s_3	s_5	s_4	s_6	2	3	4	7	6	6	5	5
s_2	$\varnothing$	$\varnothing$	$\varnothing$	s_7	s_5	s_4	$\vdots$	$\vdots$	$\vdots$	1	5	7	$\vdots$	6
s_3				$\varnothing$	s_6	$\varnothing$				6	$\vdots$	$\vdots$		$\vdots$
s_4					$\varnothing$					$\vdots$				
$\varnothing$														

We assume that school s_5 is the only first-preference-first school. We have

$$\mathrm{FPF}^3(P) = \mathrm{GS}^3(P) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \varnothing & s_1 & s_2 & s_3 & s_5 & s_4 & s_6 \end{pmatrix},$$

where every student but student 1 has received her most preferred school. Therefore, only student 1 may be able to benefit by misrepresenting her preferences to FPF^3 and GS^3. Let $P_1^{s_4}$ denote one of student 1's preference relations where she ranks only school s_4 as acceptable. Then

$$\mathrm{GS}^3\big(P_1^{s_4}, P_{-1}\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ s_4 & s_1 & s_2 & s_3 & s_7 & s_5 & s_6 \end{pmatrix}.$$

By reporting the preference relation $P_1^{s_4}$ to GS^3, student 1 is matched to an acceptable school s_4 but is unmatched when she reports her true preference relation P_1. Therefore, the profile P is vulnerable under GS^3.

However, because school s_5 is a first-preference-first school and as student 5 has ranked it higher than student 6, we have (where $\succ_{s_5}$ has been replaced by $\hat{\succ}_{s_5}$)

$$\mathrm{FPF}^3\big(P_1^{s_4}, P_{-1}\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \varnothing & s_1 & s_2 & s_3 & s_5 & s_6 & s_4 \end{pmatrix}.$$

By reporting the preference relation $P_1^{s_4}$ under FPF^3, student 1 is unmatched, same as when she reports her true preferences. It is enough to check for misrepresentation by ranking schools first.[6] In addition, student 1 cannot misrepresent her preferences to obtain a seat at school s_1, s_2 and s_3. Therefore, the preference profile P is not vulnerable under FPF^3.

The intuition is that when student 1 applies to school s_4, she causes the rejection of student 6. Then student 6 applies to school s_5. Under GS^3, student 5 is rejected from school s_5 and she applies to school s_7, ending the process. However, under FPF^3, school s_5 is a first-preference-first school which student 5 ranks first. This time it is student 6 who is rejected from school s_5. Then she applies to school s_6 and causes the rejection of student 7. Ultimately, student 7 applies to school s_4 and takes it back from student 1.

Note that this example does not contradict our result (Theorem 1). Indeed, consider a preference profile $(\hat{P}_7, P_{-7})$ where P_7 has been replaced by a preference relation

[6]This is because if for some student i and a school s, $\mathrm{GS}_i(P_i', P_{-i}^k) = s$, then by ranking school s first under P_i^s, we have $\mathrm{GS}_i(P_i^s, P_{-i}^k) = s$ (Roth 1982).

$\hat{P}_7$ where s_6 is the only acceptable school for student 7. Then $\text{FPF}^3(\hat{P}_7, P_{-7}) = \text{FPF}^3(P)$ where student 1 is unmatched. However, $\text{FPF}^3_4(P_1^{s_4}, \hat{P}_7, P_{-\{1,7\}}) = s_4$. That is, school s_4 is strategically accessible to student 1 via FPF^3. ◊

The oversight in the proof of Claim 1 by Pathak and Sönmez (2013) is that they only consider the case where $\text{FPF}^k(P)$ is not stable under P and there is a student with justified envy toward another student at a first-preference-first school. However, they did not consider that a student can have a justified envy toward another student at an equal preference school. Unfortunately, as the above example shows, this may occur. Nevertheless, when each school is a first-preference-first school the manipulability comparison of the constrained Boston mechanism and the constrained Gale–Shapley mechanism is valid (Pathak and Sönmez 2011). Table 2 features the reforms in England and Wales that can and that cannot be evaluated using the manipulability criterion.

4.2 *Limitation under homogeneous preferences*

Manipulability compares mechanisms across all preference profiles. Let us focus on specific preference profiles. First, let us look at the preference profiles where the constrained serial dictatorship mechanism is not manipulable. When the constrained serial dictatorship SD mechanism is not manipulable, its outcome is the same as the outcome of its unconstrained version.

Proposition 1. *Let $k \geq 1$. A preference profile P is not vulnerable under the constrained serial dictatorship mechanism* SD^k *if and only if* $\text{SD}^k(P) = \text{SD}(P)$.

Proof. The "if" part is straightforward. If $\text{SD}^k(P) = \text{SD}(P)$, then at $\text{SD}^k(P)$ each student is matched to her best available school among the remaining ones and cannot profitably misreport her preferences.

We prove the "only if" part by contraposition. Suppose that $\text{SD}^k(P) \neq \text{SD}(P)$ and consider the highest priority student i for whom $\text{SD}^k_i(P) \neq s = \text{SD}_i(P)$. Each student with higher priority than i received under $\text{SD}^k(P)$ the same school as under $\text{SD}(P)$. Therefore, under $\text{SD}^k(P)$ student i had the same choice set of remaining schools as under $\text{SD}(P)$. The only way i missed school s under $\text{SD}^k(P)$ is if the constraint k was binding for her, that is, each of her top k schools were already assigned, and school s was not listed among the top k acceptable schools under P_i. However, school s still had available seats, and i could manipulate SD^k at P by listing school s as one of her top k acceptable schools. □

Abdulkadiroğlu et al. (2011) argue that in many real life contexts of school choice, students tend to value schools based on the same set of qualities such as safety and academic reputation. As a result, they have similar ordinal preferences. For these profiles, the constraint is almost guaranteed to be binding for at least one student, and thus the constrained serial dictatorship SD mechanism is manipulable. In this domain, manipulability does not distinguish the constrained Boston mechanism and the constrained serial dictatorship mechanism. Next, we generalize the Chicago example presented in

the Introduction. We show that when students have tier preferences, the constrained serial dictatorship SD mechanism is manipulable.

EXAMPLE 2 (Serial dictatorship and tier preferences). We consider a school choice context with n students and m schools such that for each $s, s' \in S$, $q_s = q_{s'} = \lambda$, and $\succ_s = \succ_{s'}$. We assume that students have tier preferences. That is, the set S of schools is partitioned into $t > 1$ sets $S_1, S_2, \ldots, S_t$. For each tier $\ell < t$, each student i prefers each school in S_ℓ over each school in $S_{\ell+1}$.[7] We assume that each student finds each school acceptable and that there is a shortage of seats, that is, $n > \sum_s q_s = \lambda \times m$.

Whenever the number of schools except the schools in the lower tier is at least as large as the constraint, $|S_1| + \cdots + |S_{t-1}| \geq k$, no student ranks a school in S_t among the top k acceptable schools. The constrained serial dictatorship mechanism SD^k is manipulable at every tiered preference profile. Indeed, if every student reports her preferences truthfully, then some students are unmatched while the schools in S_t are unassigned. Under Proposition 1, this is necessary and sufficient for the manipulability of the constrained serial dictatorship mechanism SD^k. Thus, for these tier preferences, the constrained serial dictatorship mechanism is as manipulable as the constrained Boston mechanism. ◊

However, strategic accessibility distinguishes the constrained serial dictatorship mechanism and the constrained Boston mechanism on the entire domain.

PROPOSITION 2. *Let $k \geq 1$. Let the capacities of the schools be increasingly ordered $q_1 \leq q_2 \leq \cdots \leq q_{|S|}$ and $\alpha = q_1 + \cdots + q_k$. Then the constrained serial dictatorship mechanism SD^k is strategy-proof for each of the α-highest priority students.*

See the Appendix for the proof. Proposition 2 is stated for the entire domain of preference profiles, and it remains true in the domain of tier preferences. Therefore, in Example 2, the share of students for whom SD^k is strategy-proof is $\lambda \times k/n$.

The schools in the upper tiers are not strategically accessible to any student. By switching from β^4 to SD^4 in 2009, and to SD^6 in 2010, the share of students to whom the Chicago elite schools are not strategically accessible increased from 4% to 24% and eventually to 100%, respectively.

5. STRATEGIC ACCESSIBILITY IN EQUILIBRIUM

In this section, we develop a more refined concept of strategic accessibility. Previously, we called the admission of student i to school s strategic if there exists a preference profile and a profitable deviation for i that places her at s. But this deviation was not required to be optimal among all deviations. The same is true for the reports of the other students: we did not assume that these students best-respond. Now, we require the strategies to be mutually optimal.

Let us motivate this with an example.

[7] Coles et al. (2013) observed that the academic job market has this structure and referred to it as block correlated preferences.

EXAMPLE 3. We consider a school choice context with three students i, j, and k and three schools s_1, s_2, and s_3 such that each school has one seat. We specify a preference profile and priority profile as follows:

P_i	P_j	P_k	$\succ_{s\in S}$
s_1	s_1	s_1	j
s_2	s_2	s_2	k
s_3	s_3	s_3	i
$\varnothing$	$\varnothing$	$\varnothing$	

We consider the Boston mechanism. Its outcome is as follows:

$$\beta(P) = \begin{pmatrix} i & j & k \\ s_3 & s_1 & s_2 \end{pmatrix}.$$

Suppose instead that student i reports the preference relation $P_i^{s_2}$ where she ranks school s_2 first. If student j and k report truthfully as in P, we have

$$\beta\big(P_i^{s_2}, P_{-i}\big) = \begin{pmatrix} i & j & k \\ s_2 & s_1 & s_3 \end{pmatrix}.$$

According to the notion developed earlier, school s_2 is strategically accessible to student i via the Boston mechanism. However, it is not a best response for student k to report truthfully P_k, when student i reports $P_i^{s_2}$. Student i has the lowest priority at every school. This strategic accessibility of student i to school s_2 stems from the fact that the other students reported their preferences truthfully without best-responding. ◊

The type of strategic accessibility featured in the example may disappear when we require best responses. To take these best responses into account, we introduce an equilibrium concept. Any mechanism φ induces a normal form game such that the students are the players, the strategies are the preference reports, and the outcome function is φ. Then a strategy profile P' is a *Nash equilibrium* of the game $[I, P, \varphi]$ if for each student i, P'_i is a best response to P'_{-i}.[8] We simply denote the game as $[P, \varphi]$.

DEFINITION 4. Let φ be a mechanism. School s is *not strategically accessible* to student i via mechanism φ *in equilibrium* if there is no preference profile P and a preference relation $\hat{P}_i$ such that:

(1) $(\hat{P}_i, P_{-i})$ is a *Nash equilibrium* of the game $[P, \varphi]$ and

(2) $s = \varphi_i(\hat{P}_i, P_{-i})\ P_i\ \varphi_i(P)$

Otherwise, school s is strategically accessible to student i via φ in equilibrium.

We use this notion to rank the mechanisms as in Definition 2.

[8] That is, for each student j, there is no strategy P''_j such that $\varphi_j(P''_j, P'_{-j})\ P_j\ \varphi_j(P')$.

DEFINITION 5. Mechanism φ is *strongly less strategically accessible than* ψ if (i) every school which is not strategically accessible to a student via ψ *in equilibrium* is also not strategically accessible to her via φ *in equilibrium* and, (ii) there is a school choice context where a school is not strategically accessible to a student via φ *in equilibrium* but via ψ.

With this notion, we establish results in line with Theorem 1 and Theorem 2.

THEOREM 4. *(i) Let $k > 1$ and suppose that there are at least k schools and at least one first-preference-first school. Then the constrained Gale–Shapley mechanism* GS^k *is strongly less strategically accessible than the constrained First-Preference-First mechanism* FPF^k. *(ii) Let $k > \ell$ and suppose that there are at least k schools. Then the constrained Gale–Shapley mechanism* GS^k *is strongly less strategically accessible than the constrained Gale–Shapley mechanism* GS^ℓ.

See the Appendix for the proof. In contrast, the prior ranking of the Chinese mechanisms does not hold anymore.

PROPOSITION 3. *Not all Chinese mechanisms are comparable with strong strategic accessibility. In particular, there are e, e' with $e > e'$, and at least e schools such that the Chinese mechanism* $\mathrm{Ch}^{(e)}$ *is not strongly less strategically accessible than the Chinese mechanism* $\mathrm{Ch}^{(e')}$.

PROOF. We prove it with the following example. We consider a school choice context where there are four students and four schools such that each school has one seat. We specify a preference profile P^* and a priority profile $\succ$ as follows:

P_i^*	P_j^*	P_k^*	P_m^*	P_t^*	P_i'	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
s_3	s_3	s_2	s_2	s_4	s_1	k	i	j	t
s_4	s_2	s_1	s_3	$\varnothing$	s_2	m	k	$\vdots$	$\vdots$
s_1	$\varnothing$	$\varnothing$	s_1		$\varnothing$	j	$\vdots$		
s_2			$\varnothing$			t			
$\varnothing$						i			

Then we have

$$\mathrm{Ch}^{(2)}(P^*) = \begin{pmatrix} i & j & k & m & t \\ \varnothing & s_3 & s_2 & s_1 & s_4 \end{pmatrix}.$$

Suppose that student i reports the preference relation P_i'. We show that (P_i', P_{-i}^*) is a Nash equilibrium of the game $[P^*, \mathrm{Ch}^{(2)}]$. First,

$$\mathrm{Ch}^{(2)}(P_i', P_{-i}^*) = \begin{pmatrix} i & j & k & m & t \\ s_1 & s_3 & s_2 & \varnothing & s_4 \end{pmatrix}. \tag{1}$$

In this matching, every student but i and m is matched to her most preferred school under P^*. Thus, we need to check that it is a best response for student i and m. Schools s_3 and s_4 are assigned to the highest priority students. Therefore, student i cannot get a seat at each of them by reporting a preference relation other than P_i'. Let us consider now student m. In any strategy where she did not include school s_1 among the top two acceptable schools, the outcome is the matching in equation (1). Suppose that she uses a strategy P_m' where she includes school s_1 among the top two acceptable schools. Then

$$\mathrm{Ch}^{(2)}\big(P_i', P_m', P^*_{-\{i,m\}}\big) = \begin{pmatrix} i & j & k & m & t \\ s_2 & s_3 & s_1 & \varnothing & s_4 \end{pmatrix},$$

where student m remains unmatched. Therefore, students i and m do not have a profitable deviation, and thus (P_i', P^*_{-i}) is a Nash equilibrium of the game $[P^*, \mathrm{Ch}^{(2)}]$. Since

$$s_1 = \mathrm{Ch}_i^{(2)}\big(P_i', P^*_{-i}\big)\, P_i^*\, \mathrm{Ch}_i^{(2)}\big(P^*\big),$$

school s_1 is strategically accessible to student i via $\mathrm{Ch}^{(2)}$ in equilibrium.

Next, we show that school s_1 is not strategically accessible to student i via $\mathrm{Ch}^{(1)} = \beta$ in equilibrium. Suppose that for some preference profile P and P_i'', we have

$$s_1 = \mathrm{Ch}_i^{(1)}\big(P_i'', P_{-i}\big)\, P_i\, \mathrm{Ch}_i^{(1)}(P). \tag{2}$$

We show that (P_i'', P_{-i}) is not a Nash equilibrium of the game $[P, \beta]$. This will complete the proof. The Boston mechanism produces a Pareto efficient matching with respect to reported preferences (Abdulkadiroğlu and Sönmez 2003).[9] Therefore, equation (2) implies there is a student $j' \in \{j, k, m, t\}$ who is worse off at $\beta(P_i'', P_{-i})$ compared to $\beta(P)$. We consider two cases:

Case 1: Student j' is matched to her first choice school at $\beta(P)$, denoted by s. Then j' is the highest priority student among those who ranked school s first. Since j' is worse off, and thus not matched to school s at $\beta(P_i'', P_{-i})$, student i ranks school s first under P_i'' and is matched to it. Under equation (2), $s = s_1$, which contradicts the fact that student j' has higher priority than student i at $\succ_{s_1}$.

Case 2: Student j' is not matched to her first choice school at $\beta(P)$. Let $s = \beta_{j'}(P)$. Then no student ranked school s first at P. Let $P^s_{j'}$ be a preference relation where she ranks school s first. We claim that $\beta_{j'}(P_i'', P^s_{j'}, P_{-\{i,j'\}}) = s$. Suppose, to the contrary, that this is not the case. Then, student i ranks school s first at P_i'', and is the only student who ranks it first at (P_i'', P_{-i}). Thus, $s = \beta_i(P_i'', P_{-i}) = s_1$. Since student i has lower priority than student j' under $\succ_{s_1}$, $\beta_{j'}(P_i'', P^s_{j'}, P_{-\{i,j'\}}) = s$, contradicting our assumption that student j' is not matched to school s.

Therefore, student j' has a profitable deviation from (P_i'', P_{-i}), and (P_i'', P_{-i}) is not a Nash equilibrium of the game $[P, \beta]$. □

[9] A matching μ is Pareto efficient under P if there is no matching μ' such that for each student i, $\mu'(i)\, R_i\, \mu(i)$, and for some student j, $\mu'(j)\, P_j\, \mu(j)$.

Next, we formulate a second way to refine strategic accessibility. In contrast to the past definition, we deem an admission to be strategic when the specified manipulation is a best response for the student in question without requiring truth telling to be a best response for the other students.

Definition 6. Let φ be a mechanism. School s is not *best-response strategically accessible* to student i via φ if there is no preference profile P and a misreport P_i' such that (1) P_i' is a best response to P_{-i} via φ and (2) $s = \varphi_i(P_i', P_{-i})\, P_i\, \varphi_i(P)$.

The main difference with Definition 4 is that we do not require the reports of students other than i to be best responses. School s is strongly strategically accessible to student i via $\varphi \Rightarrow$ school s is best-response strategically accessible to student i via $\varphi \Rightarrow$ school s is strategically accessible to student i via φ. Then an analogue of Theorems 1–2 can be obtained using this concept.

In addition, in all the preference profiles that we constructed to prove our results in Section 3.1, the profitable misreport of the student in consideration is a best response. Therefore, the comparisons between the Chinese parallel mechanisms (Theorem 3) are valid according to this concept.

6. Conclusion

Pathak and Sönmez (2013) show that numerous school districts have recently reformed their admissions systems to address incentive concerns. Yet, the reforms do not eliminate all possibilities for manipulation. We applied a notion introduced by Bonkoungou (2018) to show that each of these reforms resulted in the adoption of a less strategically accessible mechanism. More precisely, each reform expands, by the set inclusion, the set of schools wherein each student cannot get an admission via manipulation.

We formulated two refined versions of strategic accessibility: an optimal version and an equilibrium version. If we only count optimal manipulations, not just beneficial ones, then we compare the mechanisms with the best strategically accessible schools. All of our results carry over to this version of strategic accessibility.

The equilibrium version is more restrictive: when student i manipulates a mechanism to get an admission at a school at the preference profile P, we require P and i's deviation to form an equilibrium in the game induced by the mechanism. This concept is arguably less realistic for markets where a best response is hard to expect, for example, when the market is large, but it is a standard refining criterion for smaller problems. Our main results carry over to this equilibrium version of strategic accessibility, except the reforms in China (Theorem 4, Proposition 3).

We emphasize that strategic accessibility is not necessarily the be-all and end-all for comparing school choice mechanisms. Perhaps, the ultimate concern of the policymakers and the parents is not the vulnerability itself, but rather the complexity of finding an optimal strategy. This complexity results in drawbacks, such as higher number of mismatches (justified envy) and overall dissatisfaction with the system. Surprisingly, mechanism-designers around the world seem ready to tolerate certain levels of these

drawbacks. This could explain why they maintain constrained mechanisms even though the unconstrained version of the Gale–Shapley mechanism is strategy-proof. The continued use of constraints in school choice is an open question.

Strategic accessibility may also be of interest outside the school choice context. The set of outcomes that each agent can achieve by a manipulation may quantify her incentives, and thus the vulnerability of mechanisms. It is also important to understand which basic properties make a mechanism more or less strategically accessible than another mechanism.

Appendix: Proofs

Theorem 1. *Let $k > 1$ and suppose that there are at least k schools and at least one first-preference-first school. The constrained Gale–Shapley mechanism* GS^k *is less strategically accessible than the constrained First-Preference-First mechanism* FPF^k.

Proof. We prove the theorem through the contrapositive. Suppose that school s is strategically accessible to student i via GS^k. There is a preference profile P and a preference relation $\hat{P}_i$ such that

$$s = \mathrm{GS}_i^k(\hat{P}_i, P_{-i})\, P_i\, \mathrm{GS}_i^k(P). \tag{3}$$

We prove two facts and derive a lemma that we also use in later proofs.

Fact 1. $\mathrm{GS}_i^k(P) = \varnothing$. *In equation (3) student i is better off misrepresenting her preferences at P. As shown by Pathak and Sönmez (2013),* $\mathrm{GS}_i^k(P) = \varnothing$.

Fact 2. *Student i did not rank school s among the top k acceptable schools under P_i. Otherwise, school s would have been acceptable under P_i^k and as $\mathrm{GS}_i(P^k) = \varnothing$, we would have*

$$\mathrm{GS}_i\big(\hat{P}_i^k, P_{-i}^k\big) = s\, P_i^k\, \varnothing = \mathrm{GS}_i\big(P^k\big).$$

This equation contradicts the fact that the Gale–Shapley mechanism is strategy-proof. Because GS^k is individually rational, equation (3) implies that school s is acceptable to student i under P_i. Now, because school s is acceptable under P_i but not under P_i^k, student i ranks more than k schools acceptable under P_i. We formulate this as a lemma.

Lemma 1. *Let P be a preference profile and suppose that for some student i, some school s, and some preference relation P_i', we have*

$$s = \mathrm{GS}_i^k\big(P_i', P_{-i}\big)\, P_i\, \mathrm{GS}_i^k(P).$$

Then (i) student i ranks more than k schools acceptable under P_i, and (ii) school s is acceptable under P_i but is not ranked among the top k schools under P_i.

Now, we prove that school s is strategically accessible to student i via FPF^k. According to Lemma 1, student i ranks more than k acceptable schools. Let $s_1, \ldots, s_k$ denote the acceptable schools that she has ranked first, second, through k under P_i^k. Let $\mu = \text{GS}^k(P)$. Then μ is stable under P^k. Following Fact 1, we have $\mu(i) = \varnothing$. For each $\ell \leq k$, we have $s_\ell \, P_i^k \, \mu(i)$, and by the stability of μ under P^k, $|\mu^{-1}(s_\ell)| = q_{s_\ell}$ and for each student $j \in \mu^{-1}(s_\ell)$, $j \succ_{s_\ell} i$. We consider the following preference profile P^*:

$$\begin{array}{cc} P_i^* & P_{j\neq i}^* \\ \hline s_1 & \mu(j) \\ s_2 & \varnothing \\ \vdots & \\ s_k & \\ s & \\ \varnothing & \end{array} \tag{4}$$

For each $\ell \leq k$, each of the students in $\mu^{-1}(s_\ell)$ has higher priority than i under $\succ_{s_\ell}$ and has ranked it first under P^*. Thus, $\text{FPF}_i^k(P^*) \neq s_\ell$. Therefore, $\text{FPF}^k(P^*) = \mu$ where student i is unmatched. Let P_i^s be a preference relation where student i finds only school s acceptable. We claim that $\text{FPF}_i^k(P_i^s, P_{-i}^*) = s$. We consider two cases:

Case 1: $|\mu^{-1}(s)| < q_s$. In this case, it is clear that $\text{FPF}_i^k(P_i^s, P_{-i}^*) = s$ because no more than q_s students find school s acceptable under (P_i^s, P_{-i}^{*k}).

Case 2: $|\mu^{-1}(s)| = q_s$. In this case, we claim that there is at least one student in $\mu^{-1}(s)$ who has lower priority than student i under $\succ_s$. Suppose, to the contrary, that each student in $\mu^{-1}(s)$ has higher priority than i under $\succ_s$.

Since $\mu = \text{GS}(P^k)$ is stable under P^k, it is also stable under (P_i^s, P_{-i}^k). This is because $\mu(i) = \varnothing$ and every student in $\mu^{-1}(s)$ has higher priority than i under $\succ_s$. As shown by Roth (1986), under responsive priorities, the set of students who are matched is the same at all stable matchings. Since student i is unmatched under μ, she is also unmatched under $\text{GS}(P_i^s, P_{-i}^k)$. According to a second result shown by Roth (1982), $\text{GS}_i(P_i'^k, P_{-i}^k) = s$ implies that $\text{GS}_i(P_i^s, P_{-i}^k) = s$. This contradicts the previous conclusion that $\text{GS}_i(P_i^s, P_{-i}^k) = \varnothing$. Therefore, there is at least one student in $\mu^{-1}(s)$ who has lower priority than student i under $\succ_s$.

Thus, $\text{FPF}_i^k(P_i^s, P_{-i}^*) = s$. Finally, because $\text{FPF}_i^k(P^*) = \varnothing$ and school s is acceptable under P_i^* by construction,

$$s = \text{FPF}_i^k\left(P_i^s, P_{-i}^*\right) P_i^* \, \text{FPF}_i^k\left(P^*\right),$$

proving that school s is strategically accessible to student i via FPF^k.

Finally, we construct a school choice context where a school is strategically accessible to some student via FPF^k but not via GS^k. We consider a school choice context where each school has one seat and they have a common priority order. By assumption, there is at least one first-preference-first school. Let s be one such school. Without loss of generality, suppose that student i is ordered first, j second and m third, in the common priority order. Since $k \geq 2$, for each preference profile where j ranks at least two

acceptable schools, she is always matched, under GS^k, to one of the acceptable schools that she has ranked first or second. Therefore, s is not strategically accessible to student j via GS^k. Since there are at least $k \geq 2$ schools, there is a school s' distinct from s. Let P be a preference profile such that the components for i, j and m are specified as below.

P_i	P_j	P_m
s'	s'	s
s	s	s'
$\varnothing$	$\varnothing$	$\varnothing$

Then, $\mathrm{FPF}_j^k(P) = \varnothing$ because school s is a first-preference-first school which student m has ranked first and that student j has ranked second. Let P_j^s be a preference relation where s is the only acceptable school for student j. Then, $\mathrm{FPF}_j^k(P_j^s, P_{-j}) = s$. Therefore, school s is strategically accessible to student j via FPF^k. □

THEOREM 2. *Let $k > \ell$ and suppose that there are at least k schools. The constrained Gale–Shapley mechanism GS^k is less strategically accessible than the constrained Gale–Shapley mechanism GS^ℓ.*

PROOF. We prove the theorem through the contraposition. Suppose that school s is strategically accessible to student i via GS^k. Then there is a preference profile P and a preference relation $\hat{P}_i$ such that

$$s = \mathrm{GS}_i^k(\hat{P}_i, P_{-i})\, P_i\, \mathrm{GS}_i^k(P).$$

We show that school s is strategically accessible to student i via GS^ℓ.

Under Lemma 1, student i ranks more than k schools acceptable under P_i and school s is acceptable under P_i but is not ranked among the top k schools. Let $s_1, \ldots, s_k$ denote the schools that student i has ranked first, second, through k, respectively, under P_i.

Let $\mu = \mathrm{GS}(P^k)$. Then, μ is stable under P^k. By Fact 1, $\mu(i) = \varnothing$. Thus, for each $\ell' \leq k$, because $s_{\ell'}\, P_i^k\, \mu(i)$ and μ stable under P^k, we have $|\mu^{-1}(s_{\ell'})| = q_{s_{\ell'}}$ and for each $j \in \mu^{-1}(s_{\ell'})$, $j \succ_{s_{\ell'}} i$. We consider the following preference profile P^*:

P_i^*	$P_{j\neq i}^*$
s_1	$\mu(j)$
s_2	$\varnothing$
$\vdots$	
s_ℓ	
s	
$\varnothing$	

(5)

Then, $GS^\ell(P^*) = \mu$, where student i is unmatched. Let P_i^s be a preference relation where student i ranks school s as the only acceptable school. If $|\mu^{-1}(s)| < q_s$, then $\mathrm{GS}_i^\ell(P_i^s, P_{-i}^*) = s$. If $|\mu^{-1}(s)| = q_s$, then by Case 2 above, student i has higher priority

than at least one student in $\mu^{-1}(s)$. Therefore, $\mathrm{GS}_i^{\ell}(P_i^s, P_{-i}^*) = s$. Therefore, school s is strategically accessible to student i via GS^{ℓ}.

We provide a school choice context where a school is strategically accessible to a student via GS^{ℓ} but not via GS^{k}. We consider a school choice context where each school has one seat and they have a common priority order. Then, the Gale–Shapley mechanism is the serial dictatorship mechanism. Let $1, 2, \ldots, |I|$ denote the student who is ordered first, second, through $|I|$ according to the common priority order. Because $k \geq \ell + 1$ and that there are at least k schools, Proposition 2 implies that no school is strategically accessible to student $\ell + 1$ via GS^{k}. Now, let P be a preference profile where students rank all schools acceptable and have a common ranking. Since there are more students than schools, by assumption, and at least k schools, $\mathrm{SD}_{\ell+1}^{\ell}(P) = \varnothing$ while $\mathrm{SD}_{\ell+1}^{k}(P) = s$ for some school s. Let $P_{\ell+1}^s$ be a preference relation where student $\ell + 1$ ranks s first. Then, $\mathrm{SD}_{\ell+1}^{\ell}(P_{\ell+1}^s, P_{-\{\ell+1\}}) = s$. Thus, school s is strategically accessible to student $\ell + 1$ via GS^{ℓ} but not via GS^{k}. □

Proof of Corollary 1. Suppose that school s is strategically accessible to student i via GS^{k}. According to Theorem 2, it is also strategically accessible to student i via GS^{ℓ}. Under Theorem 1, school s is also strategically accessible to student i via FPF^{ℓ}.

It remains to provide a school choice context where the converse is not true. We consider the school choice context provided in the proof of Theorem 1. There, a school s is strategically accessible to student i via FPF^{ℓ} but not GS^{ℓ}. Under Theorem 2, school s is not strategically accessible to student i via GS^{k}. □

Theorem 3. *Let $e > e'$ and suppose that there are at least e schools. The Chinese mechanism $\mathrm{Ch}^{(e)}$ is less strategically accessible than the Chinese mechanism $\mathrm{Ch}^{(e')}$.*

Proof. We collect some basic results that are established by Chen and Kesten (2017).

Lemma 2 (Chen and Kesten 2017). *Let P be a preference profile and P_i^s a preference relation where student i ranks school s first.*

(i) If $\mathrm{Ch}_i^{(e)}(P) = s$ then we have $\mathrm{Ch}_i^{(e)}(P_i^s, P_{-i}) = s$.

(ii) Suppose that student i ranks school s among her top e acceptable schools under P_i and $s\, P_i\, \mathrm{Ch}_i^{(e)}(P)$. There is no preference relation $\hat{P}_i$ such that $\mathrm{Ch}_i^{(e)}(\hat{P}_i, P_{-i}) = s$.

We prove the theorem through the contraposition. Suppose that school s is strategically accessible to student i via $\mathrm{Ch}^{(e)}$. There is a preference profile P and a preference relation $\hat{P}_i$ such that

$$s = \mathrm{Ch}_i^{(e)}(\hat{P}_i, P_{-i})\, P_i\, \mathrm{Ch}_i^{(e)}(P). \tag{6}$$

Let P_i^s be a preference relation where student i has ranked school s first. By part (i) of Lemma 2, $\mathrm{Ch}_i^{(e)}(P_i^s, P_{-i}) = s$. Then, under $\mathrm{Ch}^{(e)}(P_i^s, P_{-i})$, student i is matched in the first round of the mechanism. Thus,

$$\mathrm{GS}_i\big(P_i^s, P_{-i}^e\big) = s. \tag{7}$$

Since $s\ P_i\ \mathrm{Ch}_i^{(e)}(P)$, student i has been rejected by school s in some round. Hence, all the seats of school s have been assigned under $\mu = \mathrm{Ch}^{(e)}(P)$. That is, $|\mu^{-1}(s)| = q_s$. Let $N = \mu^{-1}(s)$.

In part (ii) of Lemma 2, equation (6) implies that student i did not rank school s among her top e acceptable schools under P. Together with equation (6), if $\mu(i)$ is a school, then it is not ranked among the top e acceptable schools under P_i. We consider two cases:

Case 1: At least one student in N has lower priority than student i under $\succ_s$. Since $e' < e$, student i has ranked more than e' schools above school s under P_i. Let $s_1, \ldots, s_{e'}$ denote the schools that student i has ranked first, second, through e', respectively, under P_i. We consider the following preference profile P^*:

P_i^*	$P_{j\neq i}^*$
s_1	$\mu(j)$
$\vdots$	$\varnothing$
$s_{e'}$	
s	
$\mu(i)$	
$\varnothing$	

Recall that student i is not matched in the first round under $\mathrm{Ch}^{(e)}(P)$. Thus $\mathrm{GS}_i(P^e) = \varnothing$. Then, for each $\ell = 1, \ldots, e'$, each student in $\mu^{-1}(s_\ell)$ has higher priority than i under $\succ_{s_\ell}$.[10] Furthermore, under P^*, each student in N ranks school s first and student i did not rank it among the top e'. Therefore,

$$\mathrm{Ch}^{(e')}\big(P^*\big) = \mu.$$

Since at least one student in N has lower priority than student i under $\succ_s$, we have

$$\mathrm{Ch}^{(e')}\big(P_i^s, P_{-i}^*\big) = s.$$

Because $s\ P_i^*\ \mu(i)$, school s is strategically accessible to student i via $\mathrm{Ch}^{(e')}$.

Case 2: Every student in N has higher priority than student i under $\succ_s$. We establish the following claim.

Claim. *At least one student in N did not rank school s among the top e acceptable schools under P.*

Proof. Suppose, to the contrary, that every student in N ranks school s among the top e acceptable schools under P. Let $\eta = \mathrm{GS}(P^e)$. Student i is not matched to one of her top e acceptable schools under $\mathrm{Ch}^{(e)}(P)$. Therefore, $\eta(i) = \varnothing$. Every student in N is matched to school s under μ and has ranked it among the top e acceptable schools under P. Then, for each $j \in N$, $\eta(j) = s$. Because every student in N has higher priority than

[10] This is because, if we let $\nu = \mathrm{GS}(P^e)$, then for each $\ell = 1, \ldots, e'$, $s_\ell\ P_i^e\ \nu(i)$ and ν is stable under P^e.

student i under $\succ_s$, η is also stable under (P_i^s, P_{-i}^e). As shown by Roth (1986), under responsive priorities, the set of matched students is the same at all stable matchings. Hence, $\mathrm{GS}_i(P_i^s, P_{-i}^e) = \varnothing$, contradicting equation (7). ◁

Since there is at least one student in N who did not rank school s among the top e acceptable schools under P and that $e' < e$, there is at least one student in N who did not rank school s among the top e' acceptable schools under P. Let j be one such student. For each $\ell = 1, \ldots, e'$, let s_ℓ^i and s_ℓ^j denote the ℓ's ranked schools, from most preferred to worst, of student i and j, respectively, under P. We consider the following preference profile P^*.

P_i^*	P_j^*	$P_{k\neq i,j}^*$
s_1^i	s_1^j	$\mu(k)$
$\vdots$	$\vdots$	$\varnothing$
$s_{e'}^i$	$s_{e'}^j$	
s	s	
$\mu(i)$	$\varnothing$	
$\varnothing$		

All the seats of each of the schools $s_1^j, \ldots, s_{e'}^j$ are assigned in the first round of $\mathrm{Ch}^{(e)}(P)$. Since student i is matched (if at all) in a round later than the first round of $\mathrm{Ch}^{(e)}(P)$, $\mu(i)$ is not one of the schools $s_1^j \ldots, s_{e'}^j$. Let $\ell = 1 \ldots, e'$. Because student j has ranked school s_ℓ^j among the top e acceptable schools under P and has been rejected, all its seats have been assigned at the first round of $\mathrm{Ch}^{(e)}(P)$ to students who have higher priority than her under $\succ_{s_\ell^j}$. Thus, each student in $\mu^{-1}(s_\ell^j)$ has higher priority than student j under $\succ_{s_\ell^j}$. Because $j \succ_s i$ and $|\mu^{-1}(s)| = q_s$, we have

$$\mathrm{Ch}^{(e')}(P^*) = \mu.$$

Now, under (P_i^s, P_{-i}^*), there are q_s students (including student i) who have ranked school s among the top e' acceptable schools. Therefore, $\mathrm{Ch}_i^{(e')}(P_i^s, P_{-i}^*) = s$ and

$$s = \mathrm{Ch}_i^{(e')}(P_i^s, P_{-i}^*)\, P_i^*\, \mathrm{Ch}_i^{(e')}(P^*),$$

proving that school s is strategically accessible to student i via $\mathrm{Ch}^{(e')}$.

We provide a school choice context where a school is strategically accessible to a student via $\mathrm{Ch}^{(e')}$ but not via $\mathrm{Ch}^{(e)}$. We consider the school choice context where schools have a common priority order and each of them has one seat. Let $1, 2, \ldots, |I|$ denote the first, second, through the last ordered student, respectively. We consider student $e' + 1$. Because there are at least e schools and that $e \geq e' + 1$, under Proposition 2, SD^e is strategy-proof to student $e' + 1$.[11] Note now that for each preference profile P,

[11] This is because student $e' + 1$ is one of the e-highest priority students under the common priority.

$\mathrm{Ch}^{(e)}_{e'+1}(P) = \mathrm{SD}^{e}_{e'+1}(P)$. Thus, $\mathrm{Ch}^{(e)}$ is strategy-proof to her. Since there are more students than schools and at least e schools, $|I| \geq e'+2$. We consider the following preference profile P

$P_{i\neq e'+2}$	$P_{e'+2}$
s_1	$s_{e'+1}$
$\vdots$	$\vdots$
$s_{|S|}$	

Since student $e'+2$ is the only student who ranked school $s_{e'+1}$ among the top e' acceptable schools, then $\mathrm{Ch}^{(e')}_{e'+2}(P) = s_{e'+1}$. In addition, the schools $s_1, \ldots, s_{e'}$ are matched to the students $1, \ldots, e'$, respectively, under $\mathrm{Ch}^{(e')}(P)$. Thus, student $e'+1$ is matched to a school (if any) that she finds worse than $s_{e'+1}$. Let $P^{s_{e'+1}}_{e'+1}$ be a preference relation where she has ranked school $s_{e'+1}$ first. Then, we have $\mathrm{Ch}^{(e)}_{e'+1}(P^{s_{e'+1}}_{e'+1}, P_{-\{e'+1\}}) = s_{e'+1}$. Therefore, school $s_{e'+1}$ is strategically accessible to her via $\mathrm{Ch}^{(e')}$ □

Proposition 2. *Let $k \geq 1$. Let the capacities of the schools be increasingly ordered $q_1 \leq q_2 \leq \cdots \leq q_{|S|}$ and $\alpha = q_1 + \cdots + q_k$. Then, the constrained serial dictatorship mechanism SD^k is strategy-proof to the α-highest priority students.*

Proof. The serial dictatorship mechanism SD is strategy-proof. Let P be a preference profile and suppose that student i is matched under $\mathrm{SD}^k(P)$ or has ranked k acceptable schools or less. Under Lemma 1, student i cannot manipulate SD^k at P. Let i be one of the α-highest priority students. We show that she never misses one of her k most preferred schools whenever she ranks at least k acceptable schools. This will complete the proof.

Suppose, to the contrary, that student i has ranked at least k acceptable schools and ends up unmatched under $\mathrm{SD}^k(P)$. Then, at her turn, all the seats of her k most preferred schools S', have been selected. Then, at least $\lambda = \sum_{s\in S'} q_s$ students have moved before her. Then, i is not one of the λ-highest priority students. This contradicts the fact that i is one of the α-highest priority students because $\alpha \leq \lambda$. □

Theorem 4. *(i) Let $k > 1$ and suppose that there are at least k schools and at least one first-preference-first school. Then, the constrained Gale–Shapley mechanism GS^k is strongly less strategically accessible than the constrained First-Preference-First mechanism FPF^k. (ii) Let $k > \ell$ and suppose that there are at least k schools. Then, the constrained Gale–Shapley mechanism GS^k is strongly less strategically accessible than the constrained Gale–Shapley mechanism GS^ℓ.*

Proof. Suppose that school s is strategically accessible to student i via GS^k in equilibrium. There is a preference profile P and a preference relation P'_i such that

- (P'_i, P_{-i}) is a Nash equilibrium of $[P, \mathrm{GS}^k]$ and
- $s = \mathrm{GS}^k_i(P'_i, P_{-i})\ P_i\ \mathrm{GS}^k_i(P)$.

We show that school s is strategically accessible to student i via FPF^k and GS^ℓ in equilibrium. The difference with the proof of Theorem 1 and Theorem 2 is that we further assumed that (P_i', P_{-i}) is a Nash equilibrium of $[P, \text{GS}^k]$. For each of the preference profiles that we constructed in equations (4) and (5), we have

$$s = \text{GS}_i^k\big(P_i^s, P_{-i}^*\big)\, P_i^*\, \text{GS}_i^k\big(P^*\big),$$

where student i has ranked school s first under P_i^s. Note that there is a student in $\mu^{-1}(s)$ who has lower priority than student i under $\succ_s$. Let j be the lowest priority students among them. Now, under $\text{FPF}^k(P_i^s, P^*)$ student j is unmatched, student i is matched to school s and each of the remaining students is matched to their first choice school. The strategy (P_i^s, P_{-i}^*) is a Nash equilibrium of $[P, \text{FPF}^k]$. Indeed, student i cannot get a seat at a school s' that she prefers to s because each student in $\mu^{-1}(s')$ has ranked s' first and has higher priority than her under $\succ_{s'}$. Similarly, each student matched to school s under $\text{FPF}^k(P_i^s, P_{-i}^*)$ has higher priority than student j under $\succ_s$ and has ranked it first under (P_i^s, P_{-i}^*). Thus, student j cannot be matched to school s by reporting a preference relation other than P_j^*. This proves that school s is strategically accessible to student i via FPF^k in equilibrium.

The argument can be used to prove that (P_i^s, P_{-i}^*) is a Nash equilibrium of the game $[P, \text{GS}^\ell]$ where P^* is the preference profile in equation (5).

We provide a school choice context where a school is strategically accessible to a student via FPF^k in equilibrium but not via GS^k. We consider a school choice context where schools have a common priority and where each school has one seat. By assumption, there are at least $k \geq 2$ schools and students. Let students be ordered from 1, the highest priority student, to $|I|$, the lowest priority student. Under Proposition 2, $\text{GS}^k = \text{SD}^k$ is strategy-proof to student 2. Since there is at least one first-preference-first school and, without loss of generality, let us assume that school s_2 is a first-preference-first school. Let P be the following preference profile:

P_1	P_2	P_3	$P_{-\{1,2,3\}}$
s_1	s_1	s_2	$\varnothing$
$\varnothing$	s_2	$\varnothing$	
	$\varnothing$		

Since $k \geq 2$, $\text{FPF}_2^k(P) = \varnothing$. Let $P_2^{s_2}$ be a preference relation where student 2 has ranked school s_2 first. Clearly, $(P_2^{s_2}, P_{-2})$ is a Nash equilibrium of the game $[P, \text{FPF}^k]$ and

$$s_2 = \text{FPF}_2^k\big(P_2^{s_2}, P_{-2}\big)\, P_2\, \text{FPF}_2^k(P).$$

Therefore, school s_2 is strategically accessible to student 2 via FPF^k in equilibrium.

We consider the same school choice context to show that a school is strategically accessible to a student via GS^ℓ in equilibrium but not via GS^k. Since $k \geq \ell + 1$, under

Proposition 2, $GS^k = SD^k$ is strategy-proof to student $\ell+1$. We consider the following preference profile P (recall that there are at least k schools).

$$\begin{array}{c} P_{i\in I} \\ \hline s_1 \\ \vdots \\ s_\ell \\ s_{\ell+1} \\ \varnothing \end{array}$$

Then, $GS^\ell_{\ell+1}(P) = \varnothing$. Let $P^{s_{\ell+1}}_{\ell+1}$ be a preference relation where student $\ell+1$ has ranked school $s_{\ell+1}$ first. Clearly, $(P^{s_{\ell+1}}_{\ell+1}, P_{-\{\ell+1\}})$ is a Nash equilibrium of the game $[P, GS^\ell]$, and

$$s_{\ell+1} = GS^\ell_{\ell+1}\big(P^{s_{\ell+1}}_{\ell+1}, P_{-\{\ell+1\}}\big)\, P_{\ell+1}\, GS^\ell_{\ell+1}(P).$$

Then, $s_{\ell+1}$ is strategically accessible to student $\ell+1$ via GS^ℓ in equilibrium. □

References

Abdulkadiroğlu, Atila, Yeon-Koo Che, and Yosuke Yasuda (2011), "Resolving conflicting preferences in school choice: The 'Boston mechanism' reconsidered." *American Economic Review*, 101, 399–410. [893]

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003), "School choice: A mechanism design approach." *American Economic Review*, 93, 729–747. [886, 887, 897]

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson (2014a), "Budget balance, fairness, and minimal manipulability." *Theoretical Economics*, 9, 753–777. [884]

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson (2014b), "Least manipulable envy-free rules in economies with indivisibilities." *Mathematical Social Sciences*, 69, 43–49. [884]

Arribillaga, R. Pablo and Jordi Massó (2016), "Comparing generalized median voter schemes according to their manipulability." *Theoretical Economics*, 11, 547–586. [884]

Bonkoungou, Somouaoga (2018), "Essays on matching and preference aggregation." [884, 898]

Bonkoungou, Somouaoga and Alexander Nesterov (2020a), "Incentives in matching markets: Counting and comparing manipulating agents." Report. [885]

Bonkoungou, Somouaoga and Alexander Nesterov (2020b), "Reforms meet fairness concerns in school and college admissions." Working paper, Higher School of Economics Research Paper No. WP BRP 234/EC/2020. [885]

Chen, Peter, Michael Egesdal, Marek Pycia, and M. Bumin Yenmez (2016), "Manipulability of stable mechanisms." *American Economic Journal: Microeconomics*, 8, 202–214. [884]

Chen, Yan and Onur Kesten (2017), "Chinese college admissions and school choice reforms: A theoretical analysis." *Journal of Political Economy*, 125, 99–139. [882, 884, 888, 890, 902]

Coles, Peter, Alexey Kushnir, and Muriel Niederle (2013), "Preference signaling in matching markets." *American Economic Journal: Microeconomics*, 5, 99–134. [894]

Decerf, Benoit and Martin Van der Linden (2020), "Manipulability in school choice." Unpublished paper, Available at SSRN 2809566. [884]

Dubins, Lester E. and David A. Freedman (1981), "Machiavelli and the Gale–Shapley algorithm." *American Mathematical Monthly*, 88, 485–494. [889]

Dur, Umut M., Parag A. Pathak, Fei Song, and Tayfun Sönmez (forthcoming), "Deduction dilemmas: The Taiwan assignment mechanism." *American Economic Journal: Microeconomics*. [884]

Gale, David and S. Shapley Lloyd (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15. [886]

Haeringer, Guillaume and Flip Klijn (2009), "Constrained school choice." *Journal of Economic Theory*, 144, 1921–1947. [888]

Kesten, Onur (2006), "On two competing mechanisms for priority-based allocation problems." *Journal of Economic Theory*, 127, 155–171. [891]

Nie, Haifeng (2007), "A game-theoretical analysis of China's college admission mechanism." *China Economic Quarterly Beijing*, 6, 899. [882]

Pathak, Parag and Tayfun Sönmez (2008), "Leveling the playing field: Sincere and sophisticated players in the Boston mechanism." *American Economic Review*, 98, 1636–1652. [882, 887]

Pathak, Parag A. and Tayfun Sönmez (2011), "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." Working paper, NBER Working Paper No. 16783. [893]

Pathak, Parag A. and Tayfun Sönmez (2013), "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." *American Economic Review*, 103, 80–106. [881, 882, 884, 885, 887, 890, 891, 893, 898, 899]

Roth, Alvin E. (1982), "The economics of matching: Stability and incentives." *Mathematics of Operations Research*, 7, 617–628. [889, 892, 900]

Roth, Alvin E. (1985), "The college admissions problem is not equivalent to the marriage problem." *Journal of Economic Theory*, 36, 277–288. [885]

Roth, Alvin E. (1986), "On the allocation of residents to rural hospitals: A general property of two-sided matching markets." *Econometrica*, 54, 425–427. [900, 904]

Co-editor Federico Echenique handled this manuscript.

Manuscript received 4 February, 2020; final version accepted 3 November, 2020; available online 24 November, 2020.