

Cooperative strategic games

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The *value* is a solution concept for n -person strategic games, developed by Nash, Shapley, and Harsanyi. The value of a game is an a priori evaluation of the economic worth of the position of each player, reflecting the players' strategic possibilities, including their ability to make threats against one another. Applications of the value in economics have been rare, at least in part because the existing definition (for games with more than two players) consists of an ad hoc scheme that does not easily lend itself to computation. This paper makes three contributions: We provide an axiomatic foundation for the value; exhibit a simple formula for its computation; and extend the value—its definition, axiomatic characterization, and computational formula—to Bayesian games. We then apply the value in simple models of corruption, oligopolistic competition, and information sharing.

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JEL CLASSIFICATION. C71, C72, C78.

1. INTRODUCTION

1.1 *The value solution*

A *strategic game* is a model for a multiperson competitive interaction. Each player chooses a strategy, and the combined choices of all the players determine a payoff to each of them. A problem of obvious interest, and with a long history in game theory, is this: How to evaluate, in advance of playing a game, the economic worth of a player's position?¹ A “value” is a general solution, that is, a method for evaluating the worth of any player in a given strategic game.

The value ought to reflect both the cooperative and the competitive aspects of the game. One may think of it as the expected payoff in a cooperative process that takes into account all the players' strategic possibilities, including their capacity to make threats against one another.

We make the simplifying assumption that *utility is transferable*, that is, that the players' payoffs are measured in units of a commodity that is freely exchangeable, like

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¹Alternatively: What would be the outcome if it were determined by an arbitrator?

money.² Therefore, it is reasonable to expect that the players will coordinate their strategic choices to maximize the sum of their payoffs, and that this maximal sum will be allocated in accordance with the threat powers of the players.

A value solution provides an a priori assessment of the cooperative allocation. Thus, it can serve as a tool for studying a variety of economic phenomena where side payments—utility transfers between the players—are made in response to explicit or implicit threats.

Shapley (1951, 1953) provided the original definition of a value for strategic games and Harsanyi (1963) suggested a modification. We believe that Harsanyi's definition is preferable. Its essential advantage is that it takes into account the potential damage of a threat not only to the threatened party but also to the party making the threat. (See Example 2.) Harsanyi calls his solution the modified Shapley value; others call it the Harsanyi–Shapley value; we call it simply *the value*.³

This paper makes three contributions: We provide an axiomatic foundation for the value; we exhibit a simple formula for its computation; and we extend the value—its definition, axiomatic characterization, and computational formula—to Bayesian games (Theorems 1 and 2.)

The axiomatic foundation delineates what assumptions must be made in order to justify use of the value solution. The formula makes it possible to compute the value much more easily than by following Harsanyi's original procedure, which is rather complex. (The procedure is described in Appendix B of our working paper Kohlberg and Neyman (2020).) And the extension to Bayesian games opens the door to applications of the value in a wide class of games that are of interest in information economics.⁴

The value solution has many potential applications. In this paper, we provide three examples: determining the economic value of a public official's authority to grant building permits; determining the impact of differential unit costs on profit sharing among colluding Cournot oligopolists; and determining the economic value of a player's information in a Bayesian game (Section 5).

1.2 The axioms for the value

We consider the following axioms: efficiency, symmetry, additivity, null player, balanced threats, and individual rationality.

Efficiency says that the sum of the values of all the players is the maximum available payoff.⁵

²If one wishes to have a concrete model of a game with transferable utilities, then one may think of a single prize, desirable by all players, and a game where each player's payoff is the probability of receiving the prize. The value of a player is then the a priori probability that a cooperative process (that may involve randomization) will allocate the prize to this player.

³The key idea underlying the Harsanyi modification is due to Nash (1953). An alternative name, then, could have been the Nash–Harsanyi–Shapley solution.

⁴It is somewhat surprising that Shapley (1953), Harsanyi (1963), and Myerson (1978) focused only on the complete information case. However, Kalai and Kalai (2013) defined their *coco value* for Bayesian games. In two-person Bayesian games, our definition of the value coincides with the coco value.

⁵Efficiency seems to be a reasonable axiom for the evaluation of a cooperative outcome. But one can imagine models where this axiom is rejected. Important examples are Ray and Vohra (1997) and Maskin (2008).

Symmetry says that two players whose payoffs are identical everywhere, and whose strategies can be switched without impacting any payoff, have the same value.

Additivity says that if the payoff to each player is the sum of her payoffs in two games that are unrelated to each other then the player's value is the sum of her values in those two games.⁶

The null-player axiom says that a player whose actions do not affect any player's payoff, and whose own payoff is identically zero, has value zero.

Individual rationality says that a player's value is at least her security level—the maximal payoff that the player can guarantee unilaterally, irrespective of the strategies of the other players.

Efficiency, symmetry, additivity, and null-player are strategic-game analogs of the classic Shapley axioms for the value of coalitional games.⁷ The individual rationality axiom is standard for strategic games. But the axiom of balanced threats is new. One way to motivate the axiom is to consider a public official, who may attempt to exploit his authority to shift rewards from one group of players to another. The axiom of balanced threats essentially stipulates that if the public official cannot shift rewards then his value is zero. In fact, the axiom requires less—it stipulates that if no player can shift rewards then every player's value is zero. We now turn to the formal definition.

1.3 The axiom of balanced threats

The minmax theorem of von Neumann says that for any two-person zero-sum game there exists a number, v , called the *minmax value* of the game, such that player 1 can guarantee to receive a payoff of at least v and player 2 can guarantee⁸ to receive a payoff of at least $-v$. Thus, the evaluation of player 1's position must be greater than or equal to v and the evaluation of player 2's position must be greater than or equal to $-v$. Since the sum of the evaluations cannot exceed zero, they must be v and $-v$, respectively.

Similarly, in a two-person constant sum game, where the sum of the payoffs of the two players is always c , there is a number v such that player 1 can guarantee to receive a payoff of at least v and player 2 can guarantee to receive a payoff of at least $c - v$; thus, the evaluation of the players' positions must be $(v, c - v)$.⁹

In a general-sum two-person game, it is less clear how to evaluate the players' positions. But in a seminal paper, Nash (1953) proposed a scheme for doing just that. While Nash's scheme applies more generally, for our purposes it is sufficient to consider the special case of games with transferable utility:

⁶Note that the rationale for this axiom does not depend on a cooperative point of view. For example, the mapping from strategic games to their Nash equilibrium payoffs, viewed as a set function, satisfies additivity.

⁷In Kohlberg and Neyman (2018), we show that the coalitional-game versions of these axioms characterize the value on a class of coalitional games called "games of threats." This is a very different result than the characterization of the value for strategic games, where the axioms are imposed directly on the strategic form.

⁸That is, player 1 (resp., 2) has a strategy that yields a payoff of at least v (resp., $-v$), regardless of the strategy chosen by her opponent.

⁹Note that in a two-person constant-sum game the two axioms of efficiency and individual rationality define a unique value solution.

Nash envisions a process of “bargaining with variable threats:” In an initial competitive stage, each player declares a “threat strategy,” to be used if negotiations break down. The players’ payoffs resulting from the deployment of these strategies constitute a “disagreement point.” In a subsequent cooperative stage, the players coordinate their strategies to maximize the sum of their payoffs, and share the gains relative to the disagreement point equally.

Nash observes that what matters in the disagreement point is only the difference between the players’ payoffs: If the disagreement point is (π_1, π_2) , then after the cooperative stage player 1’s payoff is $\pi_1 + \frac{1}{2}(s - (\pi_1 + \pi_2)) = \frac{1}{2}s + \frac{1}{2}(\pi_1 - \pi_2)$, and similarly player 2’s payoff is $\frac{1}{2}s - \frac{1}{2}(\pi_1 - \pi_2)$, where s denotes the maximal sum of the players’ payoffs in any entry of the payoff matrix. Thus, player 1 strives to maximize $\pi_1 - \pi_2$, while player 2 strives to minimize the same expression.

Nash then constructs an auxiliary (zero-sum) game by taking the difference between player 1’s and player 2’s payoffs. If δ denotes the minmax value of the auxiliary game, then players 1 and 2 can guarantee, at the end of the cooperative stage,

$$\frac{1}{2}s + \frac{1}{2}\delta \quad \text{and} \quad \frac{1}{2}s - \frac{1}{2}\delta, \quad (1)$$

respectively. The above pair of numbers is the Nash solution.¹⁰

For a numerical example, consider the following game.

EXAMPLE 1.

$$\begin{bmatrix} 1, 5 & 2, 4 \\ 0, 0 & 0, 0 \end{bmatrix}. \quad \diamond$$

The game of differences is

$$\begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix},$$

and its minmax value is zero. (Player 1 can guarantee zero by playing the bottom row, and she can obviously not guarantee any higher payoff.) Thus, $\delta = 0$ and $s = 6$; hence, by formula (1), the Nash solution is $(3, 3)$.

The Nash solution provides a compelling method for evaluating the cooperative outcome in a two-person game. The challenge, then, is to extend the solution to n -person games. Harsanyi (1963) and Myerson (1978) both approached this challenge by generalizing the scheme of “bargaining with variable threats,” but these n -person schemes are quite complex. (See Kohlberg and Neyman (2020), Appendix B.) By contrast, we focus on a basic property of the Nash solution and adopt it as an axiom that can be generalized to n -person games.

¹⁰The simple definition, by means of formula (1), for the Nash bargaining solution in games with transferable utilities is due to Shapley (1984). Kalai and Kalai (2013) independently discovered the formula and used it to define their competitive-cooperative solution concept, the *coco value*.

The basic property is this: *If $\delta \geq 0$, then the value of player 1 is greater than or equal to the value of player 2.* (This follows from equation (1).)

In order to generalize this property to n -player games, we proceed as follows. Consider, for any subset S of the set of players, N , an auxiliary two-player zero-sum game between S and its complement, $N \setminus S$, where the players in each of these subsets coordinate their strategies (and pool their information) to act as a single player, and where the payoff to player S is the difference between the sum of the (original game) payoffs to the players in S and the sum of the payoffs to the players in $N \setminus S$; and define $\delta(S)$ as the minmax value of this game.

Now, consider two players, i and j , and assume that $\delta(S) \geq 0$ for any subset S that includes i but not j . Then—by the same logic as in a two-player game,—in the cooperative outcome player i 's payoff should be greater than or equal to player j 's payoff. Therefore, if $\delta(S) = 0$ for any subset S that includes i but not j , and consequently—by the minmax theorem— $\delta(S) = 0$ for any subset S that includes j but not i , the payoffs to i and j should be equal. This, essentially, is the assumption of “balanced threats.”

In fact, to prove our uniqueness result, we require less. We assume that only if the above holds for any pair i and j , that is, if $\delta(S) = 0$ for any proper¹¹ subset of N , then the payoffs to all players should be equal. Furthermore, we weaken the axiom even more¹² by requiring the condition only in games of pure transfers, where $\delta(N)$ —the maximum sum of the players' payoffs¹³—is equal to zero. Thus, the axiom of balanced threats is defined as follows: *if $\delta(S) = 0$ for all $S \subseteq N$, then the value of each player is zero.*¹⁴

1.4 The uniqueness result and the formula for the value

Theorems 1 and 2 state our main results—that the axioms of efficiency, balanced threats, symmetry, null player, and additivity imply a unique value solution for strategic games with complete information as well as for Bayesian games, and that the value satisfies individual rationality; furthermore, the theorems provide a formula for computing the value.

The formula says that the value of a player in an n -person strategic game or Bayesian game is an average of the threat powers, $\delta(S)$, of the subsets of which the player is a member. Specifically, *if $\delta_{i,k}$ denotes the average of $\delta(S)$ over all k -player subsets that include i , then the value of player i is the average of $\delta_{i,k}$ over $k = 1, 2, \dots, n$.*

REMARK 1. In a two-player game with complete information, the value coincides with the Nash variable-threats solution. Indeed, the formula says that the value of player 1 is $\frac{1}{2}\delta(1) + \frac{1}{2}\delta(1, 2)$, which is the same as equation (1).

¹¹A proper subset of N is a subset that is neither $\emptyset$ nor N .

¹²This means that the uniqueness theorem is stronger.

¹³Note that $\delta(N)$ is the minmax value of a redundant two-person game between the all-player set N and the empty set; it is therefore natural to think of $\delta(N)$ as the maximum sum of the players' payoffs.

¹⁴In Kohlberg and Neyman (2020), Appendix D, we show that the value satisfies several additional properties, each of which can replace the axiom of balanced threats in the characterization of the value.

1.5 *The impact of inferior strategies*

We end this [Introduction](#) by emphasizing the fundamental distinction between the cooperative-competitive approach underlying the idea of a value and the purely competitive approach underlying the concept of equilibrium.

Consider once again the two-player game of [Example 1](#). In purely-competitive analysis, the strategy “down” is viewed as an *incredible threat*; thus, the availability of this strategy does not affect the equilibrium outcome, (1, 5). But in an explicit or implicit cooperative process, player 1’s threat to play “down” cannot be ignored. Indeed, the Nash solution—(3, 3)—exhibits a side payment from player 2 to player 1, as does Shapley’s original notion of value (see [Section 2](#) for details.) Other authors, for example, [Green \(2005\)](#) have also emphasized the impact of inferior strategies on the cooperative-competitive outcome.

1.6 *Organization of the paper*

[Section 2](#) discusses the historical development of the ideas. In [Sections 3](#) and [4](#), we define the axioms and state the main results—the axiomatic characterization and the formula for computing the value. In [Section 5](#), we apply the formula in a number of examples. In [Section 6](#), we present a characterization of the von Neumann–Morgenstern–Shapley value that parallels the characterization of the value, and in [Section 7](#) we provide the proofs of our main results. The [Appendix](#) shows that all the axioms are tight, that is, if any of them is dropped then the uniqueness theorem is no longer valid.

2. HISTORY OF THE CONCEPTS

This section reviews the historical development of the ideas at the foundation of the notion of value. Skipping this section will not affect the reader’s understanding of the rest of the paper.

In the classic book, [von Neumann and Morgenstern \(1953\)](#), the starting point for the cooperative analysis of strategic games is to reduce every such game to a characteristic function, nowadays called a *coalitional game*, which assigns to every subset of players (“coalition”) S a single number, $v(S)$, defined as the total payoff that the members of S can guarantee, that is, the maxmin of the sum of the payoffs to the members of S , where the max is over all the correlated strategies of S and the min is over the correlated strategies of the complement of S . Having reduced strategic games to coalitional games, vNM focused on developing their solution concept for coalitional games, the “stable set.”

In contrast to vNM’s set-valued solution, Shapley highlighted the need to define a single-valued function that assigns to each strategic game a vector of payoffs, representing the value of each role in the game. Shapley accepted the vNM approach of reducing strategic games to coalitional games; thus, he addressed the problem of defining a value function for coalitional games. In a seminal paper, [Shapley \(1953\)](#) formulated properties (“axioms”) that would be desirable in such a function and proved that—remarkably—a mere four of them uniquely imply one particular function, the “Shapley value.”

It would seem then that Shapley's goal of defining a value function for strategic games had been accomplished: given a strategic game, transform it to its vNM coalitional form, then apply the Shapley value. But there were doubts. The doubts, centering on the adequacy of the vNM coalitional game, were raised by vNM¹⁵ and Shapley themselves, as well as by Luce and Raiffa (1957), Harsanyi (1963), and Myerson (1978). As Shapley (1953) wrote: "The difficulty, intuitively, is that the characteristic function does not distinguish between threats that damage just the threatened party and threats that damage both parties."

The difficulty with the vNM coalitional game—the reason that it does not properly reflect the threat powers of the players—arises because a coalition is allowed to deploy two different strategies, one for maximizing its own payoff and the other for minimizing the complementary coalition's payoff. Consider Example 1. The vNM coalitional game is $v(1) = 1$, $v(2) = 0$, and $v(1, 2) = 6$. (The players' security levels—the maximum payoff that each one of them can guarantee regardless of the strategies of the opponent—are 1 and 0, respectively.) Note that player 1 plays Up in order to maximize her own payoff, but plays Down in order to minimize player 2's payoff.

Harsanyi (1963) proposed a modification in the definition of the value that is motivated by Nash's "bargaining with variable threats," and it is this modification that we call "the value." We describe Harsanyi's method below. (See Kohlberg and Neyman 2020, Appendix B, for a more complete description.)

Instead of considering two separate zero-sum games between a coalition S and its complement, one that focuses on the payoff to S and the other on the payoff to $N \setminus S$, we consider a single game that focuses on the *difference* between these payoffs; and we assign to each coalition S a single number, $\delta(S)$, defined as the maximal difference between the total payoffs to S and to $N \setminus S$ that the members of S can guarantee. By the minmax theorem, $\delta(S) = -\delta(N \setminus S)$.

Now δ is not a coalitional game. It may fail to satisfy the single condition required of a set function to qualify as a coalitional game, namely $\delta(\emptyset) = 0$. This condition is essential for the formula of the Shapley value, which assigns to each player i an average of her marginal contributions, including the marginal contribution $v(i) - v(\emptyset) = v(i)$. However, we show in Kohlberg and Neyman (2018) that an appropriate modification of the definition of the Shapley value applies to set functions such as δ , which satisfy the condition that $\delta(S) = -\delta(N \setminus S)$ for all $S \subseteq N$, and which we call "games of threats." The value of the strategic game is then obtained by taking the Shapley value of δ . We refer to this modification by Harsanyi of Shapley's original notion as *the value* of a strategic game.

It is easy to verify that the value coincides with the Nash variable-threats solution in two-player games and that the value coincides with the vNM–Shapley value in constant-sum games and in pure-exchange economies (Proposition 3).

¹⁵von Neumann and Morgenstern (1953) wrote: "In a general [-sum] game the advantage of one group of players need not be synonymous with the disadvantage of the others. In such a game, moves—or rather changes in strategy—may exist which are advantageous to both groups. . . . Does our approach not disregard this aspect?"

We wish to emphasize that neither Shapley's original definition of a value for strategic games nor Harsanyi's modification rest on an axiomatic foundation; indeed, the first step—that of reducing the strategic game to a coalitional form—is arbitrary.

REMARK 2. [Kalai and Kalai \(2013\)](#) characterize the same concept for two-player games as we do. (They call their concept the *coco value*.) However, their axioms differ from ours. For example, payoff dominance is one of the Kalai and Kalai axioms for two-player games, but it is unreasonable in games with more than two players. (See [Kohlberg and Neyman \(2020\)](#), Appendix A.)

We end this section with an example that demonstrates the different responses of the two notions of value to an increase in the cost of carrying out a threat. Consider once again the two-player game of [Example 1](#).

$$\begin{bmatrix} 1, 5 & 2, 4 \\ 0, 0 & 0, 0 \end{bmatrix}.$$

The vNM coalitional game is $v(1) = 1$, $v(2) = 0$ and $v(1, 2) = 6$. Thus, the vNM–Shapley value is $(3.5, 2.5)$. (In a two-person game, the Shapley value of player i is $\frac{1}{2}v(i) + \frac{1}{2}v(1, 2)$.) As we have seen, the value (i.e., the Nash solution) is $(3, 3)$.

Now consider the following variant.

EXAMPLE 2.

$$\begin{bmatrix} 1, 5 & 2, 4 \\ -1, 0 & 0, 0 \end{bmatrix}. \quad \diamond$$

The security levels of the players are unchanged. Thus, the vNM coalitional game is unchanged and the vNM–Shapley value is still $(3.5, 2.5)$. But the game of differences is now

$$\begin{bmatrix} -4 & -2 \\ -1 & 0 \end{bmatrix}$$

and its minmax value is -1 . Thus, $\delta = -1$ and $s = 6$; hence, by formula (1), the value is $(2.5, 3.5)$: The increased cost of the threat has had an impact on the value solution.

3. THE AXIOMS

A strategic game is a triple $G = (N, A, g)$, where

- $N = \{1, \dots, n\}$ is a finite set of players,
- A^i is the finite¹⁶ set of player i 's pure strategies, and $A = \prod_{i=1}^n A^i$,

¹⁶The assumption that the sets of players and strategies are finite is made for convenience. The results remain valid when the sets are infinite, provided the minmax value exists in the two-person zero-sum games defined in the sequel.

- $g^i: A \rightarrow \mathbb{R}$ is player i 's payoff function, and $g = (g^i)_{i \in N}$.

We use the same notation, g , to denote the linear extension

- $g^i: \Delta(A) \rightarrow \mathbb{R}$,

where for any set K , $\Delta(K)$ denotes the probability distributions on K , and we denote

- $A^S = \prod_{i \in S} A^i$, and
- $X^S = \Delta(A^S)$ (correlated strategies of the players in S).

We define the *direct sum* of strategic games as follows.¹⁷

DEFINITION 1. Let $G_1 = (N, A_1, g_1)$ and $G_2 = (N, A_2, g_2)$ be two strategic games. Then $G := G_1 \oplus G_2$ is the game $G = (N, A, g)$, where $A = A_1 \times A_2$ and $g(a) = g_1(a_1) + g_2(a_2)$.

REMARK 3. The game $G_1 \oplus G_2$ is a model for a competitive interaction where the same set of players simultaneously play two games that are independent, that is, where the moves in one game do not influence the other game.

REMARK 4. It is easy to verify that the operation $\oplus$ is, informally, commutative and associative.¹⁸ However, there is no natural notion of inverse. (In general, $G \oplus (-G) \neq 0$.)

Denote by $\mathbb{G}(N)$ the set of all n -player strategic games. Let $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$. This may be viewed as a map that associates with any strategic game an allocation of payoffs to the players. We consider a list of axioms for γ . To that end, we first introduce a few definitions.

Let $G \in \mathbb{G}(N)$. We define the *threat power* of coalition S as follows:¹⁹

$$(\delta G)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \left(\sum_{i \in S} g^i(x, y) - \sum_{i \notin S} g^i(x, y) \right). \quad (2)$$

We say that i and j are *interchangeable* in G if $A^i = A^j$ and $g^i = g^j$; and for any $a, b \in A^N$, if $a^i = b^j$, $a^j = b^i$, and $a^k = b^k$ for all $k \neq i, j$, then $g(a) = g(b)$.

We say that i is a *null player* in G if $g^i(a) = 0$ for all a ; and if $a^k = b^k$ for all $k \neq i$, then $g(a) = g(b)$.

We consider the following axioms. For all strategic games G ,

- *Efficiency* $\sum_{i \in N} \gamma_i G = \max_{a \in A^N} (\sum_{i \in N} g^i(a))$.
- *Balanced threats* If $(\delta G)(S) = 0$ for all $S \subseteq N$, then $\gamma_i = 0$ for all $i \in N$.
- *Symmetry* If i and j are interchangeable in G , then $\gamma_i G = \gamma_j G$.

¹⁷von Neumann and Morgenstern (1953), Section 27.6.2, refer to this operation as the *superposition* of games.

¹⁸Formally, $G_1 \oplus G_2$ is not the same game as $G_2 \oplus G_1$, because $A_1 \times A_2 \neq A_2 \times A_1$.

¹⁹Expressions of the form $\max$ or $\min$ over the empty set should always be ignored.

- *Null player* If i is a null player in G , then $\gamma_i G = 0$.
- *Additivity* $\gamma(G_1 \oplus G_2) = \gamma G_1 + \gamma G_2$.
- *Individual rationality* $\gamma_i(G) \geq \max_{x \in X^i} \min_{y \in X^{N \setminus i}} g^i(x, y)$.

REMARK 5. There are many additional desirable properties of the value that we do not assume but rather deduce from the axioms. These include dependence on the reduced form of the game (removing strategies that are convex combinations of other strategies does not affect the value), homogeneity of degree one ($\gamma(\alpha G) = \alpha \gamma G$ for $\alpha > 0$), time-consistency ($\gamma(\frac{1}{2}G_1 \oplus \frac{1}{2}G_2) = \frac{1}{2}\gamma G_1 + \frac{1}{2}\gamma G_2$; that is, it does not matter if the allocation is determined before or after the resolution of uncertainty about the game), monotonicity in actions (removing a pure strategy of a player does not increase the player's value),²⁰ independence of the set of players (addition of null players does not affect the value of the existing players), shift-invariance (adding a constant payoff to a player increases the player's value by that constant), a stronger form of symmetry (the names of the players do not matter), and continuity ($\gamma(G_n) \rightarrow \gamma G$ whenever $G_n = (N, A, g_n)$, $G = (N, A, g)$, and $g_n \rightarrow g$).

REMARK 6. We do not require, nor are we able to deduce, that $\gamma(\alpha G) = \alpha \gamma G$ for negative α . Such a requirement, which is natural in the context of coalitional games, would make no sense in the context of strategic games. The game $-G$ involves dramatically different strategic considerations than the game G , and so there is no reason to expect a simple relationship between the values of the two games.

4. THE MAIN RESULTS

4.1 The value of strategic games

Our main result for strategic games is as follows.

THEOREM 1. *There is a unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, additivity, and null player. It may be described as follows:*

$$\gamma_i G = \frac{1}{n} \sum_{k=1}^n \delta_{i,k}, \quad (3)$$

where $\delta_{i,k}$ denotes the average of $(\delta G)(S)$ over all k -player coalitions S that include i . Furthermore, this map satisfies the axiom of individual rationality.

We shall refer to the above map as the *value* for strategic games.

²⁰These four properties follow from formula (3) below and the corresponding properties of the minmax value of zero-sum games.

EXAMPLE 3. This is a three-payer game, G . Player 1 chooses the row, player 2 chooses the column, and player 3 has only a single strategy. The payoff matrix is²¹

$$\begin{bmatrix} 2, 2, 2 & 0, 0, 0 \\ 0, 0, 0 & 1, 1, 1 \end{bmatrix}.$$

Now, denoting $\delta G = \delta$,

$$\begin{aligned} \delta(1) &= \max \min \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = -\frac{2}{3}, \\ \delta(1, 3) &= \max \min \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{3}, \end{aligned}$$

$\delta(1, 2) = \max(2, 0, 1) = 2$, and $\delta(1, 2, 3) = \max(6, 0, 3) = 6$.

Thus, $\gamma_1 = \frac{1}{3}(\delta(1) + \frac{\delta(1,2)+\delta(1,3)}{2} + \delta(1, 2, 3)) = \frac{1}{3} \times (-\frac{2}{3}) + \frac{1}{3} \times \frac{2+\frac{2}{3}}{2} + \frac{1}{3} \times 6 = 2\frac{2}{9}$, and similarly $\gamma_2 = 2\frac{2}{9}$; therefore, $\gamma = (2\frac{2}{9}, 2\frac{2}{9}, 1\frac{5}{9})$. Players 1 and 2 each receive a side payment of $\frac{2}{9}$ from player 3. $\diamond$

REMARK 7. There is only one n -player coalition, namely, N . Thus, $\delta_{i,n} = (\delta G)(N)$, the maximum total payoff. The formula allocates to each player her equitable payoff, which is $\frac{1}{n}$ th of this amount, adjusted according to the average threat powers of the proper subsets that include the player.

REMARK 8. Formula (3) implies that the value of G depends only on the threats, $((\delta G)(S))_{S \subseteq N}$. We wish to emphasize that this is not an assumption but rather a conclusion. Indeed, a key step in proving the main result is the derivation of this conclusion from the axioms (Proposition 7).

REMARK 9. At first blush, it might appear that a value solution ought to satisfy the following consistency condition: If a player who is a strategic dummy (i.e., a player who has no strategic options) is dropped from the game, then the value of the remaining players remains the same. However, further reflection shows that this requirement is unwarranted: a player can exert influence on the outcome not only through her strategic choices, but also through her willingness to make side payments. (Recall that the value is an assessment of the cooperative outcome, where all players agree on the side payments.) Example 3 is a case in point: The value is $(2\frac{2}{9}, 2\frac{2}{9}, 1\frac{5}{9})$, but when player 3 (who is a strategic dummy) is dropped, the game becomes

$$\begin{bmatrix} 2, 2 & 0, 0 \\ 0, 0 & 1, 1 \end{bmatrix},$$

²¹Player 1's and player 2's payoffs are identical and their strategies can be switched without impacting any payoff. This is an example of interchangeable players.

and the value is $(2, 2)$. One can interpret the side payments of $\frac{2}{9}$ that player 3 makes to each of players 1 and 2 as incentives to not deploy the threat strategies, Down and Right.²²

4.2 The value of Bayesian games

The uniqueness theorem and formula (3) are also valid for Bayesian games. In a *Bayesian game*, each player has a finite set, C^i , of possible actions; the players do not know the “true” payoff functions, $u^i: \prod_{i \in N} C^i \rightarrow \mathbb{R}$; however, each player receives a signal, y^i , which is correlated with $u = (u^i)_{i \in N}$; specifically, the players know the “prior” probability distribution, μ , over $U \times Y$, where U and Y are the finite sets of possible payoff functions and signals, respectively.

A pure strategy for player i is now a mapping, $a^i: Y^i \rightarrow C^i$, from signals to actions, and A^i is the set of pure strategies; the payoff function, $g^i: \prod_{i \in N} A^i \rightarrow \mathbb{R}$ is the expectation $g^i(a) := E_\mu u^i(a(y))$; and a (Bayesian) correlated strategy for a subset S is a probability distribution over mappings from $\prod_{i \in S} Y^i$ to $\prod_{i \in S} C^i$. Note that in a correlated strategy the players in S not only coordinate their strategic choices, but they also pool their information.

Denote by $\mathbb{B}(N)$ the set of all n -player Bayesian games, and let $B \in \mathbb{B}(N)$. We generalize formula (2), defining the power of threat of a coalition S as follows:

$$(\delta_B G)(S) := \max_{x \in \hat{X}^S} \min_{y \in \hat{X}^{N \setminus S}} \left(\sum_{i \in S} g^i(x, y) - \sum_{i \notin S} g^i(x, y) \right),$$

where $\hat{X}^S$ denotes the set of (Bayesian) correlated strategies of S .

We can now define the axiom of balanced threats for Bayesian games in analogy with the definition for strategic games. Similarly, we define the axioms of efficiency and of individual rationality in analogy with the definitions for strategic games, replacing $a \in A^N$ by $x \in \hat{X}^N$, and $x \in X^{N \setminus i}$ by $x \in \hat{X}^{N \setminus i}$, respectively. Finally, we define the symmetry and the null-player axioms in analogy with the definitions for strategic games, adding to the definition of interchangeable players the requirement that their signals be identical, and to the definition of a null player the requirement that the player receive no signals.

THEOREM 2. *There is a unique map from $\mathbb{B}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, additivity, and null player. It is described by formula (3), modified by replacing δG with $\delta_B G$. Furthermore, this map satisfies the axiom of individual rationality.*

REMARK 10. The axiomatic characterization of the value in strategic games does not automatically follow from the characterization in Bayesian games, that is, [Theorem 1](#) is not a special case of [Theorem 2](#): in general, it is not true that if a list of axioms uniquely

²²Note, however, that if the others' choices cannot impact the dummy player's payoff, then this consideration becomes moot. Indeed, the “small worlds axiom” ([Kohlberg and Neyman \(2020\)](#), Appendix C) says that if a player has no strategic options and her payoffs are unaffected by the choices of the other players, then when the player is dropped from the game the value of the remaining players remains the same.

determines a function on a certain domain then the specialization of the same axioms to a subdomain will uniquely determine the function there.

4.3 The random-order approach

An alternative formula for computing the value is based on the random-order approach. It is analogous to the [Shapley \(1953\)](#) random-order formula for the value of a cooperative game. In some applications, it is more convenient²³ to use than formula (3).

PROPOSITION 1. *The value of a strategic game²⁴ G may be described as follows:*

$$\gamma_i G = \frac{1}{n!} \sum_{\mathcal{R}} (\delta G)(S_i^{\mathcal{R}}), \quad (4)$$

where the summation is over the $n!$ possible orderings of the set N and where $S_i^{\mathcal{R}}$ denotes the subset consisting of i and those $j \in N$ that precede i in the ordering $\mathcal{R}$.

The equivalence of formulas (3) and (4) is easy to verify.²⁵

4.4 Proofs of the theorems

The proofs of Theorems 1 and 2 are given in [Section 7.4](#). These proofs require the notion of games of threats ([Kohlberg and Neyman 2018](#)). We provide the relevant definitions and results in [Section 7.1](#).

5. APPLICATIONS OF THE VALUE SOLUTION

In this section, we describe the application of the value solution in some simple game models. More general results and their proofs will appear in forthcoming papers.

5.1 The economic worth of a public official

The examples below are highly simplified game models of a public official who has the authority to make decisions in matters of financial importance to private individuals or companies. The official's value in the game provides a measure of his potential gain from side payments, that is, bribes. Such a measure can be useful in designing systems of incentives and penalties intended to deter bribery.

5.1.1 Authority to issue licenses

EXAMPLE 4. Each one of players $i = 1, \dots, n$ seeks approval (license) for a project. Player A has the authority to approve up to k projects. This is a strategic game where player A can choose any subset of players of size at most k , while players $i = 1, \dots, n$ have no

²³See the computation of the value for the Cournot Oligopoly example in [Kohlberg and Neyman \(2020\)](#).

²⁴The proposition is valid for Bayesian games as well, provided δG is replaced by $\delta_B G$.

²⁵This equivalence is established in Proposition 3 of [Kohlberg and Neyman \(2020\)](#).

strategic choices;²⁶ player A 's payoff is identically zero, while player i 's payoff is α_i or 0, depending whether her project is approved or not. We assume that $\alpha_1 \geq \dots \geq \alpha_n > 0$. $\diamond$

One can apply formula (3) to obtain an expression for the value of this game. We describe this expression for the two extreme cases, $k = n$ and $k = 1$. In the case $k = n$, the value of player A is $\sum_{i=1}^n \alpha_i/2$. This makes intuitive sense since player A 's decision to approve any project has no bearing on her ability to approve any other project; therefore, we have, in effect, n independent 2-person games, each consisting of bargaining over the profits, α_i , from a single project.

In the case $k = 1$, the value of player A is $\sum_{i=1}^n \frac{\alpha_i}{i(i+1)}$. If all the projects are equally profitable, that is, $\alpha_i = \alpha$ for $i = 1, \dots, n$, then this is a telescopic sequence that adds up to $\alpha(1 - \frac{1}{n+1})$. Thus, when n is large player A gets essentially all the surplus. This makes intuitive sense, since the different projects are perfect substitutes for one another and, therefore, player A 's threat not to approve any specific project is extremely powerful. Thus, the player whose project does get approved, concedes most of its profit to player A .

But when the different projects have different potential profits, then player A 's threat to not approve the most profitable project is weakened by the realization that she might have to approve a less profitable project. This point is captured by the formula for the value of player A , which can be rewritten as follows: $\alpha_1 - \frac{1}{2}(\alpha_1 - \alpha_2) - \frac{1}{3}(\alpha_2 - \alpha_3) - \dots - \frac{1}{n}(\alpha_{n-1} - \alpha_n)$.

5.1.2 Authority to regulate economic activity One or more regulators has the authority to regulate an economic activity, for example, the repurposing of offices to apartments in a particular neighborhood. For simplicity, we assume that the regulation can only consist of blanket prohibition or blanket approval.

EXAMPLE 5. Each one of n players, $i = 1, \dots, n$ seeks approval for a project worth 1. Player A (a regulator) has the authority either to approve or to reject all the projects. This is a strategic game where player A has two strategies, while players $i = 1, \dots, n$ have no strategic choices; player A 's payoff is identically zero, while player i 's payoff is 1 or 0, depending on whether player A has approved or rejected the projects. $\diamond$

When n is large, it is easy to see that the value of player A is approximately $\frac{n}{4}$, while the value of each of the other players is approximately $\frac{3}{4}$. Thus, player A receives about one-fourth the total feasible output. In effect, the regulator's threat to withhold approval induces each of players $i = 1, \dots, n$ to concede one-fourth of their output to the regulator.

More generally, if there are k regulators whose approvals are required, then one can apply a continuous version of formula (3) to conclude that the payoff to each regulator divided by n converges, as $n \rightarrow \infty$, to $\int_{1/2}^1 (2x - 1)x^{k-1} dx$. In the case $k = 2$, this

²⁶As we have seen, the availability of an inferior strategy may have an impact on the value. However, in this case, giving each player $i = 1, \dots, n$ the option to not execute her project even when it has been approved, does not change the value of the game.

amounts to $\frac{5}{24}$, that is, the two regulators jointly receive about 42% of the total feasible output, compared with 25% in the case of a single regulator. Since $k \int_{1/2}^1 (2x - 1)x^{k-1} dx$ converges, as $k \rightarrow \infty$, to 1, we see that when there are many regulators—all of whose approvals are required—essentially all the economic output goes to them.

Finally, consider a variant of this game where there are k regulators, each one of whose approvals is sufficient. In this game, the payoff to each regulator divided by n converges, as $n \rightarrow \infty$, to $\frac{2^{-k}}{k(k+1)}$, and the combined payoff to all the regulators converges to $\frac{2^{-k}}{(k+1)}$. When $k = 2$, this amounts to $\frac{1}{12}$. Thus, when there are two regulators, only one of whose approvals is required, the fraction of the total value that they receive is about 8.5%, in contrast to 42% in the case where both approvals are required.

5.2 The value in a Cournot oligopoly²⁷

EXAMPLE 6. Consider a Cournot oligopoly with inverse demand function $1 - \sum_1^n q_i$, where q_i is the quantity of firm i , and with constant unit costs $c_1 < c_2 < \dots < c_n$, and assume that the firms intend to engage in a collusive arrangement. What is the profit that each firm should expect to receive? In other words, denoting the monopoly profit of firm i by $M_i := \max_q (10 - q - c_i)q$, how is the maximal available profit, M_1 , to be divided among the firms? $\diamond$

First, consider the case $n = 2$, that is, a duopoly. The Shapley value is $(\frac{M_1}{2}, \frac{M_1}{2})$. To see this, note that each firm can only guarantee zero on its own, since its rival can threaten to flood the market; therefore, $v(1) = v(2) = 0$ and $v(1, 2) = M_1$, where v denotes the vNM coalitional form of the game. Since the coalitional game is symmetric, so is its Shapley value.

But this solution does not seem to make sense—should the firm with the lower cost not receive a larger share of the profit? The reason for obtaining a “nonsensical” solution is precisely the difficulty mentioned in the [Section 2](#), namely that the Shapley value fails to take into account the damage that a threat inflicts on the party making the threat. And, obviously, the damage to a firm flooding the market is greater the greater is the firm’s unit cost.

In contrast to the Shapley value, the value solution does take account of the unit costs. The solution is $(M_1 - \frac{M_2}{2}, \frac{M_2}{2})$. Since $M_1 > M_2$, the value of firm 1 is, indeed, greater than the value of firm 2.

In the case of three firms, the value is $(M_1 - \frac{1}{2}M_2 - \frac{1}{6}M_3, \frac{1}{2}M_2 - \frac{1}{6}M_3, \frac{1}{3}M_3)$, which equals $\frac{1}{3}(M_3, M_3, M_3) + \frac{1}{2}(M_2 - M_3, M_2 - M_3, 0) + (M_1 - M_2, 0, 0)$. More generally, in the case of n firms the value may be described as follows: First, M_n , the monopoly profit of the least efficient firm, is divided equally among all the firms. Next, $M_{n-1} - M_n$ is shared equally among firms $1, \dots, n - 1$. And so on, until finally $M_1 - M_2$ is received only by firm 1.

²⁷We are grateful to an anonymous referee for suggesting that we characterize the value solution in a Cournot oligopoly.

5.3 *The value of information*

In a Bayesian game, a player can impact the side payments that she makes or receives not only through her strategic choices, but also by sharing or withholding information. Thus, the value solution, by providing an a-priori assessment of the side payments, quantifies the economic worth of information in a competitive environment. Below is a numerical example.

First, consider a two-person strategic game with complete information: Firm 2 is developing a new product and firm 1 is developing an add-on product. Each firm makes a private choice about which one of two alternative technologies it will use; and the market for the add-on product may attain one of two unknown states that are equally likely. Firm 2's profits will be the same, irrespective of the technology choices or the state of the market. (For simplicity, assume this profit is zero.) Firm 1's profit will be 4 if both firms choose the same technology and the market attains the state that is favorable to the chosen technology, and zero otherwise.

The maximal sum of (expected) payoffs in this game is 2. (The firms choose the same technology; thus, firm 1 gets the payoff 4 with probability 50%.) What is the side payment that player 2 ought to receive for its cooperation? The value solution is (1.5, 0.5), that is, the side payment is 0.5. This makes intuitive sense, as player 2 can threaten to deprive firm 1 of 50% of its payoff (of 2) by randomizing with probabilities (0.5, 0.5); thus, the magnitude of the threat is 1, and player 1 concedes one-half of this amount.

Next, consider an additional firm, 3, that specializes in market research. Firm 3 has no strategic choices but it knows which state will occur. (Note that the introduction of a player with differential information has turned the example into a Bayesian game.²⁸) Clearly, the maximal sum of payoffs is 4. But what are the side payments? The value solution is (2, 1, 1). It is interesting to note that the value of each one of the firms reflects a different consideration. Firm 1's value derives from its potential payoff of 4; firm 2's value derives from its threat to reduce player 1's payoff; and firm 3's value derives from its knowledge of the true state.

It may also be interesting to consider a variant of the game with a fourth player who, like player 3, knows the true state but has no strategic choices. The value of this game is $(2, \frac{4}{3}, \frac{1}{3}, \frac{1}{3})$: competition between the informed players has reduced their total value, to the benefit of player 2.

6. THE vNM–SHAPLEY VALUE OF STRATEGIC GAMES

In Section 2, we argued that the (von Neumann–Morgenstern–) Shapley value is a less convincing solution concept than is the (Nash–Harsanyi–Shapley) value. Here, we present a characterization of the vNM–Shapley value that parallels the characterization the value and clarifies the relationship between the two concepts; and we indicate conditions under which the concepts coincide.²⁹

²⁸Formally, the two-player game is also a game of incomplete information. However, as noted earlier, if the players' information is symmetric, then we might as well view the game as a game of complete information.

²⁹For ease of exposition, we restrict attention to strategic games with complete information, but an analogous characterization holds for Bayesian games.

6.1 Axiomatization of the vNM–Shapley value

The vNM–Shapley value of a strategic game G is the Shapley value of the vNM-coalitional game vG that is defined by

$$(vG)(S) := \max_{x \in X^S} \min_{y \in X^{N \setminus S}} \sum_{i \in S} g^i(x, y). \quad (5)$$

Let $G \in \mathbb{G}(N)$. Define

$$(\hat{\delta}G)(S) := (vG)(S) - (vG)(N \setminus S). \quad (6)$$

We introduce the following axiom. For all $G \in \mathbb{G}(N)$

Balanced security levels. If $(\hat{\delta}G)(S) = 0$ for every $S \subseteq N$, then $\hat{\gamma}_i G = 0$ for all $i \in N$.

PROPOSITION 2. *The vNM–Shapley value is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced security levels, symmetry, null player, and additivity. It may be described as follows:*

$$\hat{\gamma}_i G = \frac{1}{n} \sum_{k=1}^n \hat{\delta}_{i,k}, \quad (7)$$

where $\hat{\delta}_{i,k}$ denotes the average of $(\hat{\delta}G)(S)$ over all k -player coalitions that include i . Furthermore, this map satisfies the axiom of individual rationality.³⁰

6.2 Games where the Shapley value and the value coincide

A pure-exchange economy is a model of strategic interaction between n agents, each having an initial endowment, where each agent is free to trade with any other agent and the payoff to each agent is a function of his final allocation. Note that we have the following.

PROPOSITION 3. *In constant-sum games and in pure-exchange economies, the Shapley value and the value coincide.*

PROOF. If the strategic game is constant-sum, then an optimal strategy for S in the problem $\max_{x \in X^S} \min_{y \in X^{N \setminus S}} \sum_{i \in S} g^i(x, y)$ is also an optimal strategy for S in the problem $\max_{y \in X^{N \setminus S}} \min_{x \in X^S} \sum_{i \in N \setminus S} g^i(x, y)$. This is also true, trivially, in an exchange economy, where the sum of the payoffs to the agents in any coalition depends only on the strategies of the agents belonging to that coalition, so that any strategy for S is optimal in minimizing the total payoff to $N \setminus S$.

In both of these cases, then the minmax strategies in the two person zero-sum game where the payoff (to player 1) is the total payoff to S , are also minmax strategies in the game where the payoff is the difference between the total payoff to S and the total payoff to $N \setminus S$. Thus, the optimal values in (5) and in (2) are the same. It follows that $(\hat{\delta}G)(S) = (\delta G)(S)$, hence $\hat{\delta}_{i,k} = \delta_{i,k}$ for all $1 \leq i, k \leq n$ and, therefore, by (7) and (3), $\hat{\gamma}G = \gamma G$. $\square$

³⁰The proof of this proposition appears in [Section 7.5](#).

REMARK 11. Note that the value and the vNM–Shapley value do not coincide in exchange economies with taxes or with voting (Aumann and Kurz 1977a, 1977b, Aumann et al. 1983, 1987).

7. PROOF OF THE MAIN RESULTS

In this section, we present the proof of our main results, Theorems 1 and 2. In preparation, we provide background on games of threats and present an alternative definition of the value in terms of such games. And we present preliminary results, some of which are of interest in their own right.

7.1 Games of threats

A *coalitional game of threats* is a pair (N, d) , where

- $N = \{1, \dots, n\}$ is a finite set of players.
- $d: 2^N \rightarrow \mathbb{R}$ is a function such that $d(S) = -d(N \setminus S)$ for all $S \subseteq N$.

REMARK 12. A game of threats need not be a coalitional game as $d(\emptyset) = -d(N)$ may be nonzero.

REMARK 13. If d is a game of threats, then so is $-d$.

Denote by $\mathbb{D}(N)$ the set of all coalitional games of threats.

Let $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$. This may be viewed as a map that associates with any game of threats an allocation of payoffs to the players. Following Shapley (1953), we consider the following axioms.

For all games of threats (N, d) , (N, d_1) , (N, d_2) , and for all players i, j ,

- *Efficiency* $\sum_{i \in N} \psi_i d = d(N)$.
- *Symmetry* $\psi_i d = \psi_j d$ if i and j are interchangeable in d (i.e., if $d(S \cup i) = d(S \cup j) \forall S \subseteq N \setminus \{i, j\}$).
- *Null player* $\psi_i d = 0$ if i is a null player in d (i.e., if $d(S \cup i) = d(S) \forall S \subseteq N$).
- *Additivity* $\psi(d_1 + d_2) = \psi d_1 + \psi d_2$.

Below are two results from Kohlberg and Neyman (2018) that will be needed in the sequel.

PROPOSITION 4. *There exists a unique map $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$ satisfying the axioms of efficiency, symmetry, null player, and additivity. It may be described as follows:*

$$\psi_i d = \frac{1}{n} \sum_{k=1}^n d_{i,k}, \quad (8)$$

where $d_{i,k}$ denotes the average of $d(S)$ over all k -player coalitions that include i .

We refer to this map as the *Shapley value for games of threats*.

DEFINITION 2. Let $T \subseteq N$, $T \neq \emptyset$. The unanimity game of threats, $u_T \in \mathbb{D}(N)$, is defined by

$$u_T(S) = \begin{cases} |T| & \text{if } S \supseteq T, \\ -|T| & \text{if } S \subseteq N \setminus T, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5. Every game of threats is a linear combination of the unanimity games of threats u_T .

7.2 Rephrasing the main result

Using the notion of games of threats, we can provide an alternative definition of the value.

PROPOSITION 6. The value of a strategic game G is the Shapley value of the game of threats associated with G , that is, $\gamma = \psi \circ \delta$, where $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$, $\psi: \mathbb{D}(N) \rightarrow \mathbb{R}^n$, and $\delta: \mathbb{G}(N) \rightarrow \mathbb{D}(N)$ are as in (3), (8), and (2), respectively.

PROOF. Formula (3) is the same as formula (8), applied to the game of threats $d = \delta G$. $\square$

Thus, Theorem 1 can be rephrased as follows: $\gamma = \psi \circ \delta$ is the unique map from $\mathbb{G}(N)$ to $\mathbb{R}^n$ that satisfies the axioms of efficiency, balanced threats, symmetry, null player, and additivity.

7.3 Preliminary results

In this section, we present properties of the mapping $\delta: \mathbb{G}(N) \rightarrow \mathbb{D}(N)$ that are needed for the proof of the main result.

Let $G \in \mathbb{G}(N)$. For any $S \subseteq N$, let $(\delta G)(S)$ be as in (2).

LEMMA 1. δG is a game of threats.

PROOF. By the minmax theorem, $(\delta G)(S) = -(\delta G)(N \setminus S)$ for any $S \subseteq N$. $\square$

We refer to δG as the game of threats associated with G .

LEMMA 2. $\delta: \mathbb{G}(N) \rightarrow \mathbb{D}(N)$ satisfies:

- $\delta(G_1 \oplus G_2) = \delta G_1 + \delta G_2$ for any $G_1, G_2 \in \mathbb{G}(N)$.
- $\delta(\alpha G) = \alpha \delta G$ for any $G \in \mathbb{G}(N)$ and $\alpha \geq 0$.

PROOF. Let $\text{val}(G)$ denote the minmax value of the two-person zero-sum strategic game G . Then $\text{val}(G_1 \oplus G_2) = \text{val}(G_1) + \text{val}(G_2)$.

To see this, note that by playing an optimal strategy in G_1 as well as an optimal strategy in G_2 , each player guarantees the payoff $\text{val}(G_1) + \text{val}(G_2)$.

Now apply the above to all two-person zero-sum games played between a coalition S and its complement $N \setminus S$, as indicated in (2). $\square$

The next lemma is an immediate consequence of the definition of δ .

LEMMA 3. $\delta : \mathbb{G}(N) \rightarrow \mathbb{D}(N)$ satisfies:

- $(\delta G)(N) = \max_{a \in A^N} (\sum_{i \in N} g^i(a))$.
- If i and j are interchangeable in G then i and j are interchangeable in δG .
- If i is a null player in G , then i is a null player in δG .

Denote by $1_T \in \mathbb{R}^n$ the indicator vector of a subset $T \subseteq N$, that is, $(1_T)_i = 1$ or 0 according to whether $i \in T$ or $i \notin T$.

DEFINITION 3. Let $T \subseteq N$, $T \neq \emptyset$. The unanimity strategic game on T , henceforth the unanimity game on T , is $U_T = (N, A, g_T)$, where

$$A^i = \{0, 1\} \text{ for all } i \in N,$$

$$g_T(a) = 1_T \text{ if } a^i = 1 \text{ for all } i \in T, \text{ and } g_T(a) = 0 \text{ otherwise.}$$

That is, if all the members of T consent then they each receive 1; however, if even one member dissents, then all receive zero; the players outside T always receive zero.

LEMMA 4. Let $T \neq \emptyset$, and let $U_T \in \mathbb{G}(N)$ be the unanimity game on T and $u_T \in \mathbb{D}(N)$ be the unanimity game of threats on T . Then $\delta U_T = u_T$.

PROOF. Consider the two-person zero-sum game between S and $N \setminus S$.

If $S \cap T$ is neither $\emptyset$ nor T , then both S and $N \setminus S$ include a player in T . If these players dissent, then all players receive 0. Thus, the minmax value, $(\delta U_T)(S)$, is 0.

If $S \cap T = T$ then, by consenting, the players in S can guarantee a payoff of 1 to each player in T and 0 to all the others. Thus, $(\delta U_T)(S) = |T|$.

If $S \cap T = \emptyset$ then, by consenting, the players in $N \setminus S$ can guarantee a payoff of 1 to each player in $T \subset N \setminus S$ and 0 to all the others. Thus, $(\delta U_T)(S) = -|T|$.

By Definition 2, $\delta U_T = u_T$. $\square$

DEFINITION 4. The antiunanimity game on T is $V_T = (N, A, g)$, where $A^i = \{S \subseteq T : S \neq \emptyset\}$ and $g(S_1, \dots, S_n) = \sum_{i \in T} -1_{S_i}$.

That is, each player in T chooses a nonempty subset of T where each member loses 1. Players outside T also choose such subsets, but their choices have no impact. Thus, the payoff to any player, i , is minus the number of players in T whose chosen set includes i .

LEMMA 5. $\delta V_T = -u_T$.

PROOF. Let S be a subset of N such that $T \subseteq S$. In the zero-sum game between S and its complement, each player in S chooses a subset of T of size 1. Thus, $(\delta V_T)(S) = -|T|$.

Let S be a subset of N such that $T \cap S \neq \emptyset$ and $T \setminus S \neq \emptyset$. In the zero-sum game between S and its complement, the minmax strategies are for the players in S to choose $T \setminus S$ and for the players in $N \setminus S$ to choose $T \cap S$. The resulting payoff is $-t_1 t_2 - (-t_2 t_1) = 0$, where t_1 and t_2 are the number of elements of $T \cap S$ and $T \setminus S$, respectively. Thus, $(\delta V_T)(S) = 0$.

Therefore, $\delta V_T = -u_T$. $\square$

LEMMA 6. For every game of threats $d \in \mathbb{D}(N)$, there exists a strategic game $U \in \mathbb{G}(N)$ such that $\delta U = d$. Moreover, there exists such a game that can be expressed as a direct sum of nonnegative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the antiunanimity games $\{V_T\}_{T \subseteq N}$.

PROOF. By Proposition 5, d is a linear combination of the unanimity games of threats u_T .

$$d = \sum_T \alpha_T u_T - \sum_T \beta_T u_T \quad \text{where } \alpha_T, \beta_T \geq 0 \text{ for all } T.$$

By Lemmas 4 and 5,

$$d = \sum_T \delta(\alpha_T U_T) + \sum_T \delta(\beta_T V_T),$$

and, by Lemma 2,

$$d = \delta \left(\left(\bigoplus_{T \subseteq N} \alpha_T U_T \right) \oplus \left(\bigoplus_{T \subseteq N} \beta_T V_T \right) \right),$$

where $\oplus_T$ stands for the direct sum of the games parameterized by T . $\square$

REMARK 14. In particular, Lemma 6 establishes that the mapping $\delta: \mathbb{G}(N) \rightarrow \mathbb{D}(N)$ is onto.

As was pointed out earlier, the operation $\oplus$ does not have a natural inverse. However, we have the following.

LEMMA 7. For every $G \in \mathbb{G}(N)$, there exists a δ -inverse, that is, $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0$. Moreover, if $G' \in \mathbb{G}(N)$ is such that $\delta G' = \delta G$ then there exists $U \in \mathbb{G}(N)$ that is a δ -inverse of both G and G' .

PROOF. Consider $-\delta G \in \mathbb{D}(N)$. By Lemma 6, there exists $U \in \mathbb{G}(N)$ such that $-\delta G = \delta U$. By Lemma 2, $\delta(G \oplus U) = 0$. And if G' is such that $\delta G' = \delta G$ then, by the same argument, $\delta(G' \oplus U) = 0$. $\square$

PROPOSITION 7. *If $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$ satisfies the axioms of balanced threats, efficiency, and additivity, then γG is a function of δG .*

PROOF. Let $G, G' \in \mathbb{G}(N)$ be such that $\delta G = \delta G'$. We must show that $\gamma G = \gamma G'$. By [Lemma 7](#), there exists $U \in \mathbb{G}(N)$ such that $\delta(G \oplus U) = 0 = \delta(G' \oplus U)$. By the axiom of balanced threats, $\gamma(G \oplus U) = 0 = \gamma(G' \oplus U)$. Thus, by the additivity axiom, $\gamma G = -\gamma U = \gamma G'$. $\square$

LEMMA 8. *For any $T \neq \emptyset$ and $\alpha \geq 0$, the axioms of symmetry, null player, and efficiency determine γ on the game αU_T . Specifically, $\gamma(\alpha U_T) = \alpha 1_T$.*

PROOF. Any $i \notin T$ is a null player in U_T , and so $\gamma_i = 0$. Any $i, j \in T$ are interchangeable in U_T , and so $\gamma_i = \gamma_j$. By efficiency, the sum of the γ_i is the maximum total payoff, which since $\alpha > 0$, is $\alpha|T|$. Thus, each of the $|T|$ nonzero γ_i is equal to α . $\square$

LEMMA 9. *For any $\alpha \geq 0$, the axioms (of symmetry, null player, additivity, balanced threats, and efficiency) determine γ on the game αV_T . Specifically, $\gamma(\alpha V_T) = -\alpha 1_T$.*

PROOF. By [Lemma 8](#), the axioms determine $\gamma(\alpha U_T) = \alpha 1_T$. By [Lemmas 4](#) and [5](#), $\delta(\alpha V_T \oplus \alpha U_T) = 0$. Therefore, by the axiom of balanced threats, $\gamma(\alpha V_T \oplus \alpha U_T) = 0$. Thus, by additivity, $\gamma(\alpha V_T) = -\gamma(\alpha U_T) = -\alpha 1_T$. $\square$

REMARK 15. We cannot rely on the same proof as that of [Lemma 8](#), by appealing to symmetry and efficiency. In the game V_T , it is not true that any two players, $i, j \in T$, are interchangeable, because the payoff functions are not identical. If we had adopted a more restrictive version of the symmetry axiom—that the names of the players do not matter—then any $i, j \in T$ would be interchangeable and the direct proof would be valid. But this more restrictive version of the axiom would lead to a weaker uniqueness theorem.

PROPOSITION 8. *The map γ of formula (3) satisfies the axiom of individual rationality.*

PROOF. Let $G = (N, A, g)$ be a strategic game. By symmetry, it is sufficient to prove individual rationality for player 1, that is, that $\gamma_1 G \geq \pi^1$, where π^1 denotes player 1's security level.

Let S_1, S_2 be a partition of $N \setminus 1$. We claim that

$$(\delta G)(S_1 \cup 1) + (\delta G)(S_2 \cup 1) \geq 2\pi^1. \quad (9)$$

To see this, let $\bar{x}^1$ be a strategy that guarantees player 1 her security level, that is,

$$\min_{x^{N \setminus 1} \in X^{N \setminus 1}} g^1(\bar{x}, x^{N \setminus 1}) = \max_{x^1 \in X^1} \min_{x^{N \setminus 1} \in X^{N \setminus 1}} g^1(x^1, x^{N \setminus 1}) = \pi^1. \quad (10)$$

We have

$$\begin{aligned}
 (\delta G)(S_1 \cup 1) &= \max_{x \in X^{S_1 \cup 1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1 \cup 1} g^i(x, y) - \sum_{i \in S_2} g^i(x, y) \right) \\
 &\geq \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1 \cup 1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right) \\
 &\geq \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\pi^1 + \sum_{i \in S_1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right) \\
 &= \pi^1 + \max_{x \in X^{S_1}} \min_{y \in X^{S_2}} \left(\sum_{i \in S_1} g^i(\bar{x}^1, x, y) - \sum_{i \in S_2} g^i(\bar{x}^1, x, y) \right). \tag{11}
 \end{aligned}$$

The first inequality follows since restricting the set of available strategies cannot increase the maximum of a function, and the second inequality follows from (10) and the fact that the maxmin of a function is monotonic in that function.

Similarly, we have

$$(\delta G)(S_2 \cup 1) \geq \pi^1 + \max_{x \in X^{S_2}} \min_{y \in X^{S_1}} \left(\sum_{i \in S_2} g^i(\bar{x}^1, x, y) - \sum_{i \in S_1} g^i(\bar{x}^1, x, y) \right). \tag{12}$$

By the minmax theorem, the sum of the right-hand sides of (11) and (12) is $2\pi^1$; therefore, adding these two inequalities implies (9).

Now, as S_1 ranges over all the sets of size $k - 1$ that do not include 1, S_2 ranges over all the sets of size $n - k$ that do not include 1; thus, $S_1 \cup 1$ ranges over all the sets of size k that include 1 and $S_2 \cup 1$ ranges over all the sets of size $n - k + 1$ that include 1. Taking the average of inequality (9) over all these sets, we have

$$\delta_{1,k} + \delta_{1,n-k+1} \geq 2\pi^1,$$

where $\delta_{1,k}$ denotes the average of $(\delta G)(S)$ over all k -player coalitions that include 1.

Taking the average over $k = 1, \dots, n$, we obtain

$$2 \times \frac{1}{n} \sum_{k=1}^n \delta_{1,k} \geq 2\pi^1.$$

Thus, by formula (3), $\gamma_1 G \geq \pi^1$. □

7.4 Proof of Theorems 1 and 2

PROOF OF THEOREM 1. We first prove uniqueness. Let $G \in \mathbb{G}(N)$. Consider $\delta G \in \mathbb{D}(N)$; by Lemma 6 there exists a game $U \in \mathbb{G}(N)$ that is a direct sum of nonnegative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the antiunanimity games $\{V_T\}_{T \subseteq N}$, such that $\delta G = \delta U$.

By [Proposition 7](#), $\gamma G = \gamma U$ and so it suffices to show that γU is determined by the axioms.

Now, by [Lemmas 8 and 9](#), γ is determined on nonnegative multiples of the unanimity games $\{U_T\}_{T \subseteq N}$ and the antiunanimity games $\{V_T\}_{T \subseteq N}$. It then follows from the axiom of additivity that γ is determined on U .

To prove existence, we show that the value, $\gamma = \psi \circ \delta$, satisfies the axioms.

Efficiency, symmetry, and the null player axiom follow from [Lemma 3](#) and the corresponding properties of the Shapley value ψ .

Additivity follows from [Lemma 2](#) and the linearity of the Shapley value.

The axiom of balanced threats follows from formula (3). If $(\delta G)(S) = 0$ for all $S \subseteq N$, then $\gamma_i G = 0$ for all $i \in N$.

Finally, [Proposition 8](#) establishes that γ satisfies the axiom of individual rationality. $\square$

The proof of [Theorem 2](#) proceeds along the same lines as the proof of [Theorem 1](#), but with δ_B replacing δ .

7.5 Proof of [Proposition 2](#)

Uniqueness can be proved in the same way as in [Theorem 1](#). It is straightforward to verify that all the lemmas that involve δ remain valid when δ is replaced by $\hat{\delta}$. In particular, note that $\hat{\delta}U_T = \delta U_T = u_T$ and $\hat{\delta}V_T = \delta V_T = -u_T$.

Recall that ψ denotes the Shapley value for games of threats. The proof that $\psi \circ \hat{\delta}$ satisfies the axioms is similar to the proof in [Theorem 1](#) that $\psi \circ \delta$ satisfies the axioms of that theorem. The proof that $\hat{\gamma} = \psi \circ \hat{\delta}$ is similar to the proof of [Proposition 6](#).

APPENDIX: THE AXIOMS FOR THE VALUE ARE TIGHT

In this section, we show that the axioms for the value are tight; that is, if any one of them is dropped then the uniqueness theorem is no longer valid. Furthermore, the axioms are tight even if balanced threats and symmetry are replaced by their more restrictive versions ((BT4) of Kohlberg and Neyman (2020)) and full symmetry, respectively). Again, for ease of exposition we restrict attention to games with complete information, but analogous results are valid for Bayesian games.

Let, for all $i \in N$,

$$\gamma_i G = \frac{1}{n}(\delta G)(N), \quad (13)$$

that is, each player receives the equitable allocation. It is easy to verify that we have the following.

CLAIM 1. *The mapping $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$ defined by (13) satisfies all the axioms except for the null-player axiom.*

Let, for all $i \in N$,

$$\gamma_i G = 0. \quad (14)$$

It is easy to verify the following.

CLAIM 2. *The mapping $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$ defined by (14) satisfies all the axioms except for efficiency.*

For each integer $1 \leq k \leq n$, let π_k be the order $k, k+1, \dots, n, 1, \dots, k-1$, and let, for all $i \in N$,

$$\gamma_i G = \frac{1}{2n} \sum_{k=1}^n ((\delta G)(\mathcal{P}_i^{\pi_k} \cup i) - (\delta G)(\mathcal{P}_i^{\pi_k})), \quad (15)$$

where $\mathcal{P}_i^{\pi_k}$ consists of all players j that precede i in the order π_k .

CLAIM 3. *The mapping $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$ defined in (15) satisfies all the axioms except for symmetry.*

PROOF. It is easy to verify that the axioms of null player, balanced threats, and additivity are satisfied. As for efficiency, it is sufficient to verify it for G such that δG is a unanimity game in $\mathbb{D}(N)$.

Let then δG be the unanimity game on T , that is, $(\delta G)(S) = |T|$ if $S \supseteq T$, $-|T|$ if $S \subseteq N \setminus T$, and zero otherwise.

For $i \in T$, $(\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = |T|$ if $\mathcal{P}_i^{\pi_k} \cup i \supseteq T$, that is, if in the order π_k , i is the last among the members of T , and zero otherwise. Thus,

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^n (\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^n \sum_{i \in T} (\delta G)(\mathcal{P}_i^{\pi_k} \cup i) = \frac{1}{n} \sum_{k=1}^n |T| = |T|,$$

where the third equality follows from the fact that in each order π_k exactly one $i \in T$ is last among the members of T .

Similarly, for $i \in T$, $(\delta G)(\mathcal{P}_i^{\pi_k}) = -|T|$ if $\mathcal{P}_i^{\pi_k} \subseteq N \setminus T$, that is, if in the order π_k , i is the first among the members of T , and zero otherwise. Since in each order π_k exactly one $i \in T$ is first among the members of T , we have

$$\sum_{i \in T} \frac{1}{n} \sum_{k=1}^n (\delta G)(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^n \sum_{i \in T} (\delta G)(\mathcal{P}_i^{\pi_k}) = \frac{1}{n} \sum_{k=1}^n (-|T|) = -|T|.$$

By (15), $\sum_{i \in T} \gamma_i = \frac{1}{2}(|T| + |T|) = |T|$.

For $i \notin T$, $\mathcal{P}_i^{\pi_k} \cup i \subseteq T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq T$, and $\mathcal{P}_i^{\pi_k} \cup i \subseteq N \setminus T$ if and only if $\mathcal{P}_i^{\pi_k} \subseteq N \setminus T$. By (15), then $\gamma_i G = 0$.

Thus, $\sum_{i=1}^n \gamma_i = |T| = (\delta G)(N)$, completing the proof of efficiency.

To see that γ of equation (15) does not satisfy the symmetry axiom, consider the unanimity game on $\{1, 2, 5\}$ in the game with player set $\{1, \dots, 5\}$.

Player 1 is first in T for the order π_1 and last in T for the order π_2 . Thus, $\gamma_1 = \frac{1}{10}(|T| - (-|T|)) = \frac{|T|}{5}$.

Player 2 is first in T for the order π_2 and last in T for the orders π_3 , π_4 , and π_5 . Thus, $\gamma_2 = \frac{1}{10}(|T| - (-3|T|)) = \frac{2|T|}{5}$.

But 1 and 2 are interchangeable. $\square$

Next, observe that the vNM–Shapley value for strategic games satisfies all the axioms except for the axiom of balanced threats (See Section 6).

Finally, consider the following map. All dummy players in G receive the same as in the value formula (3), and the others share equally the remainder relative to $(\delta G)(N)$. It is easy to verify that this solution satisfies all the axioms except for additivity. (It does not even satisfy consensus-shift invariance.)

We conclude this section by commenting on the axioms required to imply that the value, γG , is a function of δG . The axiom of balanced threats says that if $(\delta G)(S) = 0$ for any subset S then $\gamma G = 0$. It would seem then that this axiom alone would suffice. However, this is not the case.

A solution that obeys the axiom of balanced threats, symmetry, efficiency, and the null player axiom but is not a function of δG can be constructed as follows.

Let δ and vG be as defined in (2) and (5), respectively, and fix $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that $f(0, y) = f(x, 0) = 0$ and $f(x, x) = x \forall x, y$.

Define $\gamma(G)$ as the Shapley value of the coalitional game u with $u(S) := f((\delta G)(S), (vG)(S))$.

CLAIM 4. *The mapping $\gamma: \mathbb{G}(N) \rightarrow \mathbb{R}^n$ defined above satisfies the axioms of balanced threats, symmetry, efficiency, and null player, but it is not a function of δG .*

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