

Characterization and uniqueness of equilibrium in competitive insurance

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This paper provides a complete characterization of equilibria in a game-theoretic version of [Rothschild and Stiglitz's \(1976\)](#) model of competitive insurance. I allow for stochastic contract offers by insurance firms and show that a unique symmetric equilibrium always exists. Exact conditions under which the equilibrium involves mixed strategies are provided. The mixed equilibrium features (i) cross-subsidization across risk levels, (ii) dependence of offers on the risk distribution, and (iii) price dispersion generated by firm randomization over offers.

KEYWORDS. Asymmetric and private information, mechanism design, oligopoly, economics of contracts, insurance.

JEL CLASSIFICATION. C72, D43, D82, D86, G22.

1. INTRODUCTION

This paper provides a complete characterization of equilibria in a game-theoretic version of [Rothschild and Stiglitz's \(1976\)](#) (henceforth, RS) model of competitive insurance with private information. I allow for stochastic contract offers by insurance firms and show that a unique symmetric equilibrium always exists, extending the classical result of RS to mixed strategies. The unique equilibrium is explicitly presented and its comparative static results are discussed. The equilibrium simultaneously features (i) cross-subsidization across risk levels, (ii) dependence of offers on the risk distribution, and (iii) price dispersion.

The literature on competitive insurance mostly restricts attention to equilibria with deterministic contract offers.¹ This restriction is problematic, as it rules out important economic phenomena present in insurance markets.

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¹See [Rothschild and Stiglitz \(1976\)](#), [Dubey and Geanakoplos \(2002\)](#), [Dubey et al. \(2005\)](#), [Bisin and Gottardi \(2006\)](#), [Guerrieri et al. \(2010\)](#).

First, the focus on deterministic contract offers implies that a static equilibrium cannot feature cross-subsidization.² Cross-subsidization means that firms may make profits from low-risk agents so as to subsidize high-risk agents. In a deterministic equilibrium, any such set of contracts is vulnerable to cream-skimming deviations by one of the competing firms, which only attract low-risk agents and leave the high risks to its competitors. However, the construction of such cream-skimming deviations hinges on firms knowing exactly which offer they are competing against, which is not true when firms use mixed strategies. This observation is relevant for policy analysis. The fact that cross-subsidization might be welfare improving has been used as a justification for government intervention (see [Bisin and Gottardi 2006](#)). In the model considered in this paper, cross-subsidization may arise in equilibrium without governmental intervention.

Second, the absence of cross-subsidization means that the contract consumed by each risk type is priced at an actuarially fair rate. Hence equilibrium contracts are independent of the relative share of each risk in the market. However, the dependence of market outcomes on risk distribution is a central theme in the policy arena.³ The equilibrium characterized in this paper is continuous with respect to the risk distribution. In particular, when almost all agents share the same risk level, the contracts chosen by consumers on-path are very similar to the full information outcomes with high probability. Under full information, the competitive model proposed in this paper leads to efficient outcomes: the consumer obtains full insurance at actuarially fair prices. In other words, our characterization is in line with the statement that markets with small frictions generate approximately efficient outcomes. This property is not present in several models that resolve the existence issue presented in RS (such as [Dubey and Geanakoplos 2002](#) and [Guerrieri et al. 2010](#)).

We consider a competitive market where firms offer contract menus to an agent who is privately informed about his own risk level. I follow [Dasgupta and Maskin \(1986\)](#) in modeling competition as a simultaneous offers game with a finite number of firms. The consumer (or agent) has private information about having high or low risk of an accident. [Dasgupta and Maskin \(1986, Theorem 5\)](#) proved the existence of equilibria for this game, but provided only a partial characterization and present no results regarding multiplicity of equilibria. The main contributions of this paper are (i) to establish uniqueness of symmetric equilibria, (ii) to solve explicitly for this equilibrium, and (iii) to derive properties and comparative statics of the equilibrium.

In [Section 4](#), I explicitly describe an equilibrium for all prior distributions. Equilibrium offers lie on a critical set of separating offers that generate zero expected profits

²This claim refers exclusively to static models of competitive insurance. In seminal papers, [Wilson \(1977\)](#), [Miyazaki \(1977\)](#), and [Riley \(1979\)](#) obtain equilibria with cross-subsidization while considering equilibrium notions incorporating anticipatory behavior akin to dynamic models.

³During the implementation of the health care exchanges following the approval of the Affordable Care Act in the United States, the presence of young adults with lower risk level was considered a necessary condition for the successful rollout and “stability” of the program (for example, see [Levitt et al. 2013](#)). In regulated markets such as the exchanges, observable conditions such as age and previous diagnostics are treated as private information since they can affect the coverage choice of consumers while not being used (or having limited use) explicitly in pricing contracts.

in the market as a whole, referred to as *cross-subsidizing offers*. The equilibrium offers coincide with the zero cross-subsidization offers described in RS whenever a pure strategy equilibrium exists. An equilibrium in pure strategies exists whenever cross-subsidization cannot lead to Pareto improvements. This occurs whenever the probability of high risks is sufficiently high⁴ (Corollary 1).

Equilibria necessarily involve mixed strategies whenever the RS menu of contracts cannot be sustained as an equilibrium. If the RS separating contracts fail to be an equilibrium outcome, the equilibrium involves each firm offering cross-subsidizing offers, with a random amount of cross-subsidization between zero and a Pareto efficient (positive) level. Offers in the support of equilibrium strategies have the following properties: (i) high-risk agents always receive a full insurance contract; (ii) low-risk agents always receive partial insurance, which leaves the high-risk agent indifferent between this contract and his own; (iii) all the menus of contracts in the support of the equilibrium strategy are ordered by attractiveness. The firm that delivers the most attractive menu of contracts attracts the customer, no matter what his type is. Moreover, firms always earn zero expected profits.⁵

The equilibrium distribution over the possible levels of cross-subsidization comes from a local condition that guarantees that, for any menu offer in the support of the equilibrium strategy, there is no local profitable deviation. I show that this condition implies there is no global profitable deviation by a firm.

In Section 5, I show that the equilibrium described is the unique symmetric equilibrium. Equilibrium offers can be described by the utility vector they generate to both possible risk types. Describing offers in terms of utility profiles means that the offer space is essentially two dimensional. The main challenge in the analysis lies in showing that equilibrium offers necessarily lie in a one-dimensional subset of the feasible utility space. The crucial step uses properties of the equilibrium utility distribution to show that expected profits are supermodular in the utility vector offered to the consumer, i.e., there is a complementarity in making more attractive offers to both risk types. This property is used to show that equilibrium offers are necessarily ordered in terms of attractiveness, i.e., a more attractive offer provides higher utility to both risk types.

Since firms make zero profits, this ordering of offers implies they generate zero profits even if they are accepted by both risk types with probability 1. Hence, offers can be indexed by the amount of subsidization that occurs across different risk types. The use of supermodularity and the zero profits condition to reduce the dimensionality of the equilibrium support is nonstandard in the literature.

⁴The competitive equilibrium concept considered in RS is different from the (game-theoretic) equilibrium concept considered here. Nevertheless, the RS pair of contracts is an equilibrium outcome of my game if and only if it is an equilibrium of their model, provided that entering firms are allowed to propose a pair of contracts. In their main definition of competitive equilibrium (Section I.4), outside firms are only allowed to offer a single contract, while it is acknowledged that a new pair of contracts might be more profitable than a deviating pooling contract (Section II.3). As a consequence, the exact condition for existence of a pure strategy equilibrium is related to separating, and not pooling, offers.

⁵In fact, each firm earns zero expected profits for any realization of its opponents' randomization (but in expectation with respect to the agent's type).

Our uniqueness result allows us to discuss comparative static exercises in a meaningful way. I analyze two relevant comparative statics exercises: with respect to the prior distribution and with respect to the number of firms. With respect to the prior distribution over types, equilibrium offers have monotone comparative statics. If the probability of low-risk agents increases, firms make more attractive offers in the sense of first-order stochastic dominance and both agent types are better off.

The equilibrium features mixing whenever cross-subsidization benefits all consumers in the market but the extent of cross-subsidies is disciplined by the absence of profitable cream-skimming deviations. On one hand, cream-skimming deviations are less attractive to high-risk types and hence attract them with smaller probability (*risk selection*). On the other hand, cream-skimming deviations introduce more risk in the consumption profile of low-risk agents (*risk inefficiency*) and hence deliver any expected utility level with lower profits for the offering firm. An increase in the share of low-risk agents in the population reduces the gain from risk selection and increases the losses from risk inefficiency. As a consequence, more cross-subsidization arises in equilibrium, which benefits consumers.

As mentioned before, the equilibrium outcome is continuous with respect to the risk distribution, even around the full information limits. When the probability of low-risk agents converges to 1, the distribution of offers converges to a mass point at the actuarially fair full insurance allocation of the low-risk agent. For such distributions, the small share of high-risk agents in the population implies that the potential gains from cream-skimming deviations are small as well. Hence a very small amount of contract uncertainty is enough to deter such deviations. Alternatively, when the probability of low-risk agent is sufficiently small, the RS pair of contracts is an equilibrium. Hence all consumers obtain actuarially fair contracts given their risk level, with high-risk consumers obtaining full coverage and low-risk agents obtaining partial coverage. Obviously, as the probability of high-risks converges to 1, the consumed contract on-path converges in probability to the full insurance contract consumed by low-risk consumers.

Equilibrium strategies also feature monotone comparative statics with respect to the number of firms, $N \geq 2$. The support of the equilibrium strategies does not change with the number of firms, but the distribution does. Surprisingly, the welfare of both types decreases with the number of firms. Each firm's offers converge to the worst pair of offers in the support: the pair of RS separating contracts. The distribution of the best offer in the market converges, as $N \rightarrow \infty$, to the equilibrium offer of a single firm in a duopoly. This result clarifies the impossibility of construction of mixed equilibrium when there are infinitely many firms and sheds light on the nonexistence results for the competitive equilibrium concept considered in RS. All comparative statics results hold with strict inequalities whenever the equilibrium involves mixed strategies.

Our model provides a rationale for the existence of cross-subsidization and has relevant empirical predictions for markets with adverse selection. First, insurance firms can be ranked in terms of attractiveness, with more attractive firms offering better contract choices for all risk types. Second, our analysis shows how risk distribution can affect market offers, increasing the presence of cross-subsidization in a welfare increasing

way. Finally, we show how an increase in the level of competition, which can be interpreted as a larger number of firms, can have adverse effects in the presence of adverse selection.

The paper is organized as follows. The next section discusses the related literature. [Section 3](#) describes the model. [Section 4](#) constructs a specific symmetric strategy profile and shows that it is an equilibrium. [Section 5](#) shows that the constructed equilibrium is the unique symmetric one. [Section 6](#) presents the comparative static results. Finally, [Section 7](#) concludes.

2. RELATED LITERATURE

Several papers have considered alternative models or equilibrium concepts that deal with the nonexistence problem in the RS model. [Maskin and Tirole \(1992\)](#) consider two alternative models: the model of an informed principal and a competitive model in which many uninformed firms offer mechanisms to the agent. In the informed principal model, the agent proposes a mechanism to the (uninformed) firm. The equilibrium set consists of all incentive compatible allocations that Pareto dominate the RS allocation. The equilibrium outcome in my model is contained in the equilibrium set of the informed principal model.

[Maskin and Tirole \(1992\)](#) also consider a competitive screening model in which firms simultaneously offer mechanisms to a privately informed agent. A mechanism is a game form in which both the chosen firm and the agent choose actions. The equilibrium set of this model is always large: it contains any allocations that are incentive compatible and satisfy individual rationality for the agent and firms. Hence the equilibrium set also contains the unique equilibrium outcome of my model.⁶

More recently, several models of adverse selection with price taking firms have been studied. [Bisin and Gottardi \(2006\)](#), [Dubey and Geanakoplos \(2002\)](#), and [Dubey et al. \(2005\)](#) consider general equilibrium models with adverse selection that always have a unique equilibrium, which has the same outcome as RS.

[Guerrieri et al. \(2010\)](#) consider a competitive search model in which the chance of an agent getting a given insurance contract depends on the ratio of insurance firms offering and agents demanding it. They show that equilibrium always exists, can be tractably characterized, and reduces to the Rothschild and Stiglitz contracts in this framework. Hence their model does not present the key empirical predictions discussed in this paper, which hinge on the presence of cross-subsidization. Since their equilibrium prediction is prior-independent, the equilibrium correspondence has a discontinuity at the perfect information case in which all agents have low risk and efficient provision of insurance occurs. In the mixed strategy equilibria described here, the probability of attracting any type of consumer varies continuously with the menu offered by a firm. This is in sharp contrast to competitive search models: if the contract consumed by low-risk

⁶The distinguishing feature of their model is the richness of the strategy set. A firm can react to moves by its opponents by offering a mechanism that contains a subsequent move by it. In equilibrium, a firm can respond to a “cream-skimming” attempt by an opponent by choosing to offer no insurance if the mechanism allows for such a move by the firm.

agents is perturbed to feature slightly more coverage, it is expected to be consumed exclusively by high-risk agents (a belief restriction is imposed on off-path contracts).

In the early contributions of [Wilson \(1977\)](#), [Miyazaki \(1977\)](#), and [Riley \(1979\)](#), it was shown that cross-subsidization can arise without randomization if firms are allowed to change their contract offers as a response to a deviation by a competitor. The equilibrium notions proposed in these papers always exist and also lead to equilibrium outcomes that are prior-dependent. These equilibrium notions have been found to be equivalent to equilibria of specific extensive forms with multiple stages by [Engers and Fernandez \(1987\)](#), [Hellwig \(1987\)](#), [Mimra and Wambach \(2011, 2016\)](#), and [Netzer and Scheuer \(2014\)](#). While the introduction of dynamics leads to interesting equilibrium outcomes with cross-subsidies, the objective of this paper is to highlight that the introduction of stochastic offers by itself leads to the same economic insights in a static framework.

[Rosenthal and Weiss \(1984\)](#) present an analysis of a competitive version of the Spence model that shares several common feature with ours. They characterize a mixed equilibrium of the model whenever a pure equilibrium does not exist. They have no results regarding uniqueness and dependence on the prior distribution. The effect of the number of firms on the constructed equilibrium is discussed, and is very similar the one presented here. [Chari et al. \(2014\)](#) characterize a mixed strategy equilibrium in a linear competitive screening model where firms are privately informed about their asset qualities.⁷

3. MODEL

A single agent faces uncertainty regarding his future income. There are two possible states $\{0, 1\}$ and his income in state 0 (1) is $y_0 = 0$ ($y_1 = 1$). The agent has private information regarding his risk type, which determines the probability of each state. For an agent of type $t \in \{h, l\}$, the probability of state 0 is p_t . Assume that $0 < p_l < p_h < 1$. This means that the l -type (low-risk) agent has higher expected income than h -type (high-risk) agents. The prior probability of type t is denoted μ_t . Define $\bar{p} \equiv \mu_l p_l + \mu_h p_h$. There are N identical firms $i = 1, \dots, N$ that compete in offering menus of contracts.

I assume that the realization of the state is contractible. A contract is a vector $c = (c_0, c_1) \in \mathbb{R}_+^2$ that specifies, respectively, the consumption level for the agent in case of low or high realized income. Contracts are exclusive. A menu of contracts is a compact subset of $\mathbb{R}_+^2$ that is denoted $\mathcal{M}^i$. The set of all compact subsets of $\mathbb{R}_+^2$ is defined as $\mathbf{M} \subseteq 2^{\mathbb{R}_+^2}$. A special case of a menu of contracts is a pair of contracts. I show later that one can focus without loss on pairs of contracts, with each one of them targeted for one specific type.

Timing is as follows. All firms simultaneously offer menus of contracts $\mathcal{M}^i \in \mathbf{M}$. Nature draws the agent's type according to probabilities μ_l and μ_h . After observing his own type t and the complete set of contracts $\mathcal{M}^1, \dots, \mathcal{M}^N$, the agent announces a choice

⁷They obtain a partial uniqueness result under the assumption that contract offers are ordered in terms of attractiveness'. In my paper, this property is a central part of my analysis as it allows one to move from a two-dimensional strategy space to a one-dimensional subset.

$a \in \bigcup_i (\mathcal{M}^i \times \{i\}) \cup \{\emptyset\}$. A choice $a = (c, i)$ indicates that contract $c \in \mathcal{M}^i$ is chosen from firm i , while choice $a = \emptyset$ means that the agent chooses to get no contract (and maintains his own income).

A final outcome of the game is $(\mathcal{M}^1, \dots, \mathcal{M}^N, t, a)$ (everything is evaluated before the income realization is revealed). Given outcome $(\mathcal{M}^1, \dots, \mathcal{M}^N, t, (c, i))$, the realized profit by firm j is zero if $j \neq i$ and otherwise is

$$\Pi(c | t) \equiv (1 - p_t)(1 - c_1) - p_t c_0.$$

The agents have instantaneous utility function $u(\cdot)$, which is strictly concave, increasing, and continuously differentiable. Finally, the utility achieved by the agent is

$$U(c | t) \equiv (1 - p_t)u(c_1) + p_t u(c_0).$$

Given outcome $(\mathcal{M}^1, \dots, \mathcal{M}^N, t, \emptyset)$, the realized profit by all firms is zero and the utility achieved by the agent is $U(y | t)$.

A (pure) strategy profile is a menu of contracts for each firm $(\mathcal{M}^i)_i$ and a choice strategy for the agent, which is a measurable function $s : \{h, l\} \times (\times_i \mathbf{M}) \rightarrow (\mathbb{R}_+^2 \times \{1, \dots, N\}) \cup \emptyset$ such that $s(t, (\mathcal{M}^i)_i) \in \bigcup_i (\mathcal{M}^i \times \{i\}) \cup \{\emptyset\}$.

A mixed acceptance rule is a Markov kernel⁸

$$s : \{h, l\} \times (\times_i \mathbf{M}) \rightarrow \Delta[(\mathbb{R}_+^2 \times \{1, \dots, N\}) \cup \emptyset],$$

with the restriction $s(\bigcup_i (\mathcal{M}^i \times \{i\}) \cup \{\emptyset\} | t, (\mathcal{M}^i)_i) = 1$ (with abuse of notation). Whenever the acceptance rule has a singleton support, we also use $s(t, (\mathcal{M}^i)_i)$ to refer to the chosen contract.

A mixed strategy profile is a probability measure over menus of contracts Φ_i for each firm i and a mixed acceptance rule s .⁹ A mixed strategy profile defines a probability distribution over outcomes in the natural way; expected profits are defined by integrating realized profits across outcomes according to this probability distribution.

The equilibrium concept is subgame perfect equilibrium.¹⁰ This means that (i) each firm i maximizes expected profits, given the strategies used by its opponents and the acceptance rule used by the agent, and (ii) the agent only chooses contracts that maximize

⁸The Markov kernel definition includes the requirement that, for any measurable set $A \subseteq \mathbb{R}_+^2 \times \{1, \dots, N\}$, the function $(t, \mathcal{M}^1, \dots, \mathcal{M}^N) \mapsto s(A | t, \mathcal{M}^1, \dots, \mathcal{M}^N)$ is measurable.

⁹I endow $\mathbf{M}$ with the Borel sigma algebra induced by the open balls in the Hausdorff metric. I use only two properties from this sigma algebra: (i) it contains any single contract and (ii) the function that leads to the best available utility to any fixed risk type must be measurable.

¹⁰In this game, perfect Bayesian equilibrium (PBE) is outcome equivalent to subgame perfection. Considering a game tree in which the firms act sequentially, each subgame perfect equilibrium has a corresponding PBE with the same strategy profiles and firms' beliefs (about the earlier firms' play) given directly by equilibrium strategies. Notice that the agent, who moves last, has perfect information because he knows all the offers and his type as well.

In this game, the concept of Nash equilibrium allows the agent to behave "irrationally" to menu offers off the equilibrium path. This enables many additional "collusive" equilibria. In fact, I can sustain any individually rational allocation as a Nash equilibrium outcome.

his (interim) utility, i.e.,

$$(c, i) \in \text{supp}(s(t, (\mathcal{M}^i)_i)) \Rightarrow c \in \arg \max_{c \in \bigcup_i \mathcal{M}^i \cup \{y\}} U(c | t)$$

and

$$\emptyset \in \text{supp}(s(t, (\mathcal{M}^i)_i)) \Rightarrow U(y | t) \geq \max_{c \in \bigcup_i \mathcal{M}^i} U(c | t).$$

The optimization problem faced by the agent always has a solution because the set of available contracts, $\bigcup_i \mathcal{M}^i \cup \{y\}$, is compact.

4. EQUILIBRIUM CONSTRUCTION

In this section, I construct an equilibrium of the described model. The existence issue raised in [Rothschild and Stiglitz \(1976\)](#) is overcome by the use of mixed strategies by insurance firms. This is the first characterization of a mixed strategies equilibrium in an insurance setting. The novel feature of this equilibrium is the potential presence of cross-subsidization, which generates a dependence of the equilibrium allocation on the risk distribution in this market. In [Section 5](#), this equilibrium is shown to be unique. In the first part of this section, I assume that $N = 2$. I show, in the end of this section, how to adjust the equilibrium to the case $N > 2$.

4.1 *Equilibrium offers*

In equilibrium, firms “compete away” profit opportunities. However, zero expected profits are consistent with cross-subsidization from low-risk to high-risk agents: the presence of losses generated from high-risk individuals, which in turn get subsidized by profits from low-risk individuals. In what follows, I construct a family of offers, indexed by the amount of cross-subsidization across types, and show that an equilibrium using these offers always exists.

Low-risk agents have higher expected income and, as a consequence, receive more attractive offers from firms. The only way to respect incentive constraints is by offering partial insurance contracts (i.e., with $c_1 > c_0$) to low-risk agents. High-risk agents, alternatively, receive less attractive contracts that do not conflict with incentive constraints. As a consequence, they receive full insurance contracts (i.e., $c_1 = c_0$). This implies that the set of contracts that arise in equilibrium lies in a restricted locus, which is described in the following paragraph.

For a level $k \in [0, p_h - \bar{p}]$ of subsidies received by high-risk individuals, the full insurance contract received by high-risk agents has consumption $c = 1 - p_h + k$, which is above their actuarially fair consumption level by k . Also define $\gamma(k) = (\gamma_1(k), \gamma_0(k))$ to be the partial insurance contracts that can be offered to the low-risk agent together with subsidy k to the high-risk agent. These contracts leave the high-risk agent indifferent between partial and full insurance, which provides incentives efficiently, and generate zero expected profits.

Formally, I define the (set-valued) function $\gamma : [0, p_h - \bar{p}] \rightarrow 2^{\mathbb{R}_+^2}$ by

$$\gamma(k) \equiv \{c \in \mathbb{R}_{++}^2 \mid U(c \mid h) = u(1 - p_h + k); \mu_l \Pi(c \mid l) + \mu_h(-k) = 0; c_1 \geq c_0\}.$$

LEMMA 1. *For any $k \in [0, p_h - \bar{p}]$, $\gamma(k)$ is a singleton, i.e., there exists a unique $c \in \mathbb{R}_+^2$ such that $c \in \gamma(k)$.*

PROOF. Let us define $\zeta = \sup\{c_1 \mid \exists c_0 \text{ such that } U(c \mid h) = u(1 - p_h + k)\}$. The strict concavity of u implies that $\zeta > 1$ ($\zeta = \infty$ is possible). Consider the path $\iota : I = [0, w] \rightarrow \mathbb{R}^2$ that starts at $(1 - p_h + k)(1, 1)$ and moves along the indifference curve of $U(\cdot \mid h)$ by increasing c_1 , i.e., $\iota_1(t) = 1 - p_h + k + t$ (and define $w \equiv \zeta - k - 1 + p_h$). Let total profit generated by point t in the path, when the high-risk agent consumes $(1 - p_h + k)(1, 1)$ and the low-risk agent consumes $\iota(t)$, be denoted as $\pi(t)$. We know that $\pi(0) \geq 0$ because $k \leq p_h - \bar{p}$. If $\zeta < \infty$, continuity implies that

$$\pi(w) \leq \mu_l(1 - p_l)(1 - \zeta) < 0.$$

If $\zeta = \infty$, it follows that $\lim_{t \rightarrow \infty} \pi(t) = -\infty$. Therefore, in both cases continuity of $\pi(t)$ implies that there is t_0 such that $\pi(t_0) = 0$. It also follows from concavity of $u(\cdot)$ that $\pi'(t) < 0$ for all $t > 0$, which means that $\pi(t_0) = 0$ for at most one point t_0 . $\square$

From now on, I refer to $\gamma(\cdot)$ as a single-valued function. Figure 1 illustrates the locus of $\{\gamma(k) \mid k \in [p_l, \bar{p}]\}$. From now on, I refer to offers

$$\mathcal{M}^k \equiv \{(1 - p_h + k, 1 - p_h + k), \gamma(k)\}$$

for $k \in [0, p_h - \bar{p}]$ as *cross-subsidizing offers*. Also define the utilities obtained from cross-subsidization level k as

$$U_l(k) \equiv U(\gamma(k) \mid l),$$

$$U_h(k) \equiv u(1 - p_h + k).$$

The pair of contracts with zero cross-subsidization coincides with the unique equilibrium allocation in RS. Given their importance on the analysis of this model, we introduce notation to refer to these contracts.

DEFINITION 1. The Rothschild–Stiglitz (RS) contracts are the pair

$$\{c_l^{\text{RS}}, c_h^{\text{RS}}\} \equiv \mathcal{M}^0.$$

We also define $u_t^{\text{RS}} \equiv U_t(0)$ for $t = l, h$ and $u^{\text{RS}} \equiv (u_l^{\text{RS}}, u_h^{\text{RS}})$.

The Pareto efficiency of cross-subsidization plays a crucial role in equilibrium analysis. Cross-subsidization always benefits high-risk agents, since their complete coverage comes at lower prices. What is more surprising is that low-risk agents can also benefit from cross-subsidization when the prior probability of high-risk is sufficiently low. The reason for that it is that subsidizing high-risk is cheap when the probability of such a

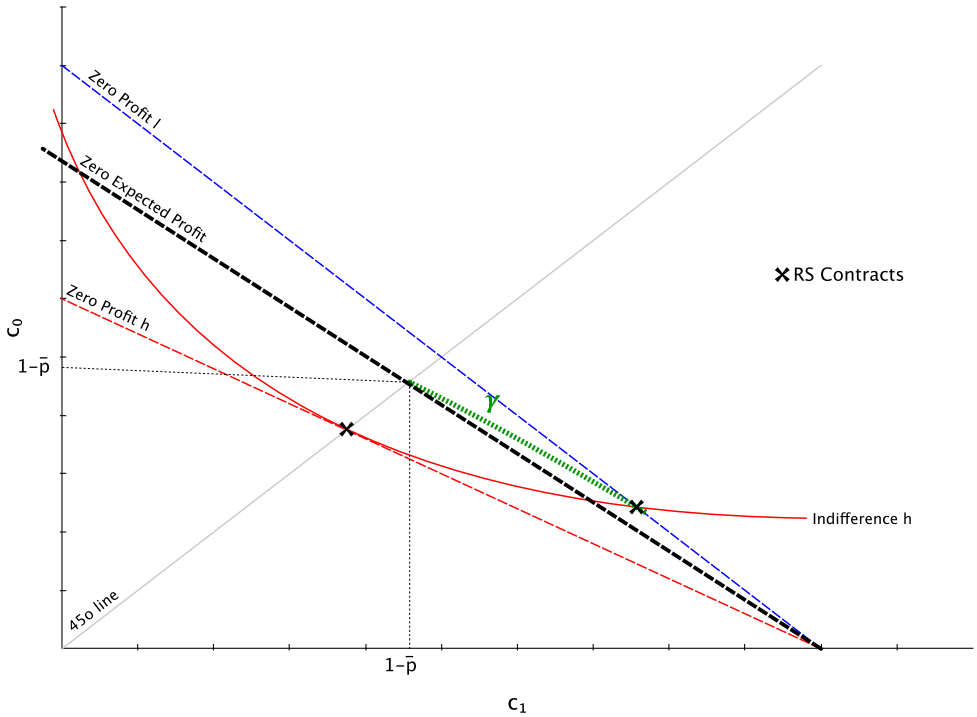


FIGURE 1. The contract space and the image of the $\gamma(\cdot)$ function. Notice that the RS partial insurance contract, c_l^{RS} , is equal to $\gamma(0)$ and coincides with the lowest level of cross-subsidization. The other extreme point in the image of $\gamma(\cdot)$ is $\gamma(p_h - \bar{p})$, which features full insurance at the correct price for the average population risk.

state is small. In the following lemma, we show that the gains from cross-subsidization are negative for large subsidization levels and are potentially positive for low subsidization levels.

LEMMA 2. *There exists $\bar{k} \in [0, p_h - \bar{p})$ such that $U_l(\cdot)$ is strictly increasing for $k < \bar{k}$ and strictly decreasing for $k > \bar{k}$. More specifically, $\bar{k}$ is the unique peak of $U_l(\cdot)$ in $[0, p_h - \bar{p}]$.*

PROOF. The implicit function theorem implies that γ is continuously differentiable and satisfies

$$\gamma'_1 = - \frac{u'(1 - p_h + k) + \frac{p_h}{p_l} u'(\gamma_0(k)) \frac{\mu_h}{\mu_l}}{p_h \left[\frac{(1 - p_l)}{p_l} u'(\gamma_0(k)) - \frac{(1 - p_h)}{p_h} u'(\gamma_1(k)) \right]},$$

$$\gamma'_0 = \frac{(1 - p_l)}{p_l} \left\{ \frac{u'(1 - p_h + k) + \frac{\mu_h}{\mu_l} \frac{(1 - p_h)}{(1 - p_l)} u'(\gamma_1(k))}{p_h \left[\frac{(1 - p_l)}{p_l} u'(\gamma_0(k)) - \frac{(1 - p_h)}{p_h} u'(\gamma_1(k)) \right]} \right\},$$

which implies that $\gamma'_1(k) < 0$ and $\gamma'_0(k) > 0$. Simple differentiation gives us

$$U'_l(k) = \left\{ \left([u'(\gamma_1(k))^{-1} - u'(\gamma_0(k))^{-1}] (1 - p_l) u'(1 - p_h + k) - \frac{\mu_h}{\mu_l} \frac{1}{p_h} \left[\frac{(1 - p_l)}{p_l} - \frac{(1 - p_h)}{p_h} \right] \right) / \left(p_h \left[\frac{(1 - p_l)}{p_l} \frac{1}{u'(\gamma_1(k))} - \frac{(1 - p_h)}{p_h} \frac{1}{u'(\gamma_0(k))} \right] \right) \right\}.$$

The numerator in the last expression is strictly positive. The denominator is strictly decreasing for $k \in [0, p_h - \bar{p}]$ and negative for $k = p_h - \bar{p}$ (in which case, $\gamma_1(p_h - \bar{p}) = \gamma_1(p_h - \bar{p}) = 1 - \bar{p}$). $\square$

Whenever cross-subsidization leads to interim efficiency gains, which happens when $\bar{k} > 0$, quasi-concavity of $U_l(\cdot)$ implies that the low-risk utility increases with the level of subsidization at any level $k \in [0, \bar{k}]$. We define this restricted set of subsidization levels, which have the property that low-risk agents benefit from it, as

$$R \equiv [0, \bar{k}].$$

A comment on the literature is in order. The Pareto efficient cross-subsidization level $\bar{k}$ defines the separating allocation that maximizes the utility offered to the low-risk agent subject to zero profits. It coincides with the allocation described by Miyazaki (1977), who obtains this allocation as an equilibrium outcome when using a reactive equilibrium notion. As described in Section 4.2, the equilibrium described here generates all cross-subsidization levels between zero and the efficient one.

If cross-subsidization is not optimal ($\bar{k} = 0$), the no cross-subsidization contracts presented in RS are indeed an equilibrium. This follows from the fact that there is no way that a firm can attract both types of agents and make expected positive profits. However, when cross-subsidization leads to interim gains ($\bar{k} > 0$), then it has to arise in equilibrium. When facing contracts $\mathcal{M}^0$ with no cross-subsidization, a firm can offer a menu with optimal subsidization $\mathcal{M}^{\bar{k}}$ that will generate zero expected profits while leading to a strictly positive utility gain for both risk types. This means that a slightly less attractive offer can make positive profits.

The presence of cross-subsidizing contract offers in a pure strategy equilibrium is ruled out because they are vulnerable to cream-skimming deviations. If firms make positive profits from low-risk agents and take losses on high-risk agents, one firm can offer a contract with slightly less coverage that attracts only low-risk agents while leaving the losses from high-risk agents to its competitors. The construction of such deviations, however, only applies to pure strategies, as a firm facing a nondegenerate distribution of competing contracts is not able to design a local deviation that attracts the low-risk agents with probability 1 while attracting high-risk agents with probability 0. In fact, we show here that firms randomize continuously between a Pareto efficient level of cross-subsidization $\bar{k}$ and the RS contracts, which feature zero cross-subsidization.

Formally, every firm mixes over menu offers:

$$\mathbf{M}^R \equiv \{\mathcal{M}^k \mid k \in R\}.$$

The set $\mathbf{M}^R$ has the property that if all firms offer menus within this set, then all offers in this set guarantee zero profits to a firm. This occurs because the firm that makes the offer $\mathcal{M}^k$ with the highest level of cross-subsidization attracts the agent, regardless of his risk type. However, the defining property of cross-subsidizing offers is that they make zero expected profits if both types consume the same menu. Firms that offer a cross-subsidizing level below their opponents make zero profits, as they never serve the agent. It is worth noting that the contract chosen by a high-risk agent in a cross-subsidizing offer always has complete coverage, only varying in the premium charged. The contract chosen by a low-risk agent, however, has degree of coverage increasing with the amount of cross-subsidization.

4.2 Equilibrium distribution

The goal of this section is to describe the equilibrium distribution over cross-subsidization levels $k \in R$, which is denoted as F , and to show that firms have no profitable deviations outside of $\mathbf{M}^R$.

The strategy set of firms contains all possible contract menus and, hence, is very large. The first step in our analysis is to describe menu offers by the expected utility it generates to each risk type. This description is useful for the following reason. For each type, the utility generated by a menu determines the probability with which it is chosen. This is the probability that the best alternative offer is less attractive than the menu considered, which is determined in equilibrium. Also, within the set of menus that deliver a specific utility profile, firms will only offer the one that minimizes expected profits. This allows us to focus on a subset of menus that are indexed by utility profiles. The profit, or loss, that is made from each type in case he joins a firm is determined by a cost minimization problem that considers the utility vectors as constraints.

Define Y as the set of incentive feasible utility profiles.¹¹ For each risk type $t \in \{l, h\}$ and utility profile $\mathbf{u} = (u_l, u_h) \in Y$, define the ex post profit function $P_t(\mathbf{u})$ as the solution to the problem

$$\max_{c \in \mathbb{R}_+^2} \Pi(c \mid t)$$

subject to generating utility u_t to type t ,

$$U(c \mid t) = u_t,$$

and respecting incentive constraints regarding type $t' \neq t$,

$$U(c \mid t') \leq u_{t'}.$$

¹¹The following notation is important for the proof. Let

$$Y \equiv \{(u_l, u_h) \in \mathbb{R}^2 \mid U(c_h \mid l) \leq U(c_l \mid l) = u_l; U(c_l \mid h) \leq U(c_h \mid h) = u_h \text{ for some } c_l, c_h \in \mathbb{R}_+^2\}.$$

Also, define as $\chi(\mathbf{u}) = (\chi_l(\mathbf{u}), \chi_h(\mathbf{u}))$ the unique solution to problems $P_l(\mathbf{u})$ and $P_h(\mathbf{u})$, respectively.

Contracts that maximize ex post profits are the most profitable ones that deliver a specific utility profile. In a mixed strategy equilibrium, the utility offered by a contract determines the probability it is chosen. Hence, there are other menus that attract both types with the same probability and generate more profits. The following properties of the optimal ex post profit function are key to our results.

LEMMA 3 (Ex post profit characterization). *The ex post profit function has the following properties:*

(i) *It is continuously differentiable.*

(ii) *Utility is costly: $\frac{\partial P_t(u)}{\partial u_t} < 0$.*

(iii) *Separation is costly:*

$$\frac{\partial P_t(u)}{\partial u_{t'}} > 0, \quad \text{if } u_t > u_{t'}$$

and

$$\frac{\partial P_t(u)}{\partial u_{t'}} = 0, \quad \text{if } u_t \leq u_{t'}.$$

(iv) *Supermodularity: $\frac{\partial P_t}{\partial u_t}$ is continuously differentiable and $\frac{\partial^2 P_t(u)}{\partial u_t \partial u_{t'}} \geq 0$, and the inequality is strict if $u_t > u_{t'}$.*

The proof is given in the [Appendix](#).

From the point of view of a single firm, the offers made by other firms can be treated as a stochastic type-contingent outside option to the agent. The distribution of outside options for a given firm is determined by the equilibrium contract distribution in the following way. On the support of equilibrium offers, higher subsidization benefits both types, so the distribution is a monotone transformation of the distribution over cross-subsidization level $k \in R$:

$$G_l(U_l(k)) \equiv F(k)$$

and

$$G_h(U_h(k)) \equiv F(k).$$

Also let $G_t(u) = 0$ for $u < U_t(1 - p_h)$ and $G_t(u) = 1$ for $u > U_t(\bar{k})$.

The distributions G_l and G_h constructed in this section are absolutely continuous. In [Section 5](#), it is shown that this is necessarily the case in equilibrium. Expected profits are determined by the probability of attracting each type, which is given by distributions $(G_t)_{t=l,h}$, and the ex post profits made from each type, which is determined by the function $(P_t)_{t=l,h}$. Now we define this function formally: for any $u = (u_l, u_h) \in Y$,

$$\pi(u) \equiv \mu_l G_l(u_l) P_l(u) + \mu_h G_h(u_h) P_h(u).$$

When contemplating a more attractive offer to a specific type, firms have to consider the following trade-off. When increasing the utility promised to such agent, he will be attracted with higher probability, which entails a gain if the firm makes profits out of such agent. However, to make a more attractive offer the firm has to make less profits out of this agent, if indeed he ends up accepting the offer. In equilibrium, firms can only make positive profits from low-risk agents. For any utility pair $u = (u_l, u_h)$ with $u_l \geq u_h$, we define the marginal profit from attracting the low-risk agent as

$$M(u) \equiv \frac{\partial \pi(u)}{\partial u_l} = \mu_l \left[g_l(u_l) P_l(u) + G_l(u_l) \frac{\partial P_l(u)}{\partial u_l} \right], \quad (1)$$

where $g_l(u_l) \equiv G'_l(u_l)$ is the density of the equilibrium utility distribution.

For cross-subsidizing offers $\mathcal{M}^k$ for $k \in R$ to arise in equilibrium, it must be optimal for a firm to offer utility profile

$$U(k) = (U_l(k), U_h(k))$$

for any $k \in R$.

This means that G_l has to satisfy the equality

$$M(U(k)) = 0 \quad \text{for all } k \in R. \quad (2)$$

Hence, local deviations around any offer in the support should not be optimal. This is a necessary condition to sustain an equilibrium with this support. Using the equality $f(k) = g_l(U_l(k))U'_l(k)$, we define F as the solution to the differential equation implied by (2).

The necessary condition for an equilibrium with support $\mathbf{M}^R$ is for F to satisfy

$$\frac{f(k)}{F(k)} = \frac{-\frac{\partial P_l(U(k))}{\partial u_l} U'_l(k)}{P_l(U(k))}, \quad (3)$$

with final condition $F(\bar{k}) = 1$.

LEMMA 4. *The differential equation (3) has a unique solution. Moreover, F is given by*

$$F(k) = \exp \left[- \int_k^{\bar{k}} \phi(z) dz \right],$$

where

$$\phi(z) = \frac{-\frac{\partial P_l(U(z))}{\partial u_l} U'_l(z)}{P_l(U(z))}.$$

Moreover, F puts no mass at zero if $\bar{k} > 0$, i.e.,

$$F(0) = 0.$$

PROOF. Integration of (3) implies that

$$1 = F(\bar{k}) = F(k) \exp \left[\int_k^{\bar{k}} \phi(z) dz \right].$$

Finally notice that $P_l(U(k)) = \Pi(\gamma(k) \mid l) = \frac{\mu_h}{\mu_l} k$. This means that $\phi(z)$ is on the order of $\frac{1}{k}$ around $1 - p_h$.¹² Then it follows that

$$\lim_{k \rightarrow (1-p_h)^+} \int_k^{\bar{k}} \phi(z) dz = \infty.$$

This implies that $F(0) = \lim_{k \rightarrow (0)^+} F(k) = 0$. □

As mentioned, condition (3) implies that any offer $\mathcal{M}^k$ is locally optimal for $k \in R$. However, in equilibrium firms also consider nonlocal deviations. To rule out such deviations, we show that the expected profits are supermodular in the utility pair offered to the agent. This means that increasing the utility offered to the high-risk agent makes it more profitable, at the margin, to make a higher utility offer to the low-risk agent.

LEMMA 5 (Supermodularity of profits). *For any $u = (u_l, u_h)$ such that $u_l \geq u_h$, we have that*

$$M(u_l, u_h) \text{ is nondecreasing in } u_h$$

and it is strictly increasing if $u_l > u_h$ and $G_l(u_l) > 0$.

The proof follows directly from the definition of $M(\cdot)$ in (1), and properties (iii) and (iv) of Lemma 3.

The relevance of the supermodularity conditions is as follows. In equilibrium, the firm must find it equally optimal to offer levels of subsidization $k, k' \in R$ with $k' > k$, which generate utilities

$$U(k) = (U_l(k), U_h(k)) \ll (U_l(k'), U_h(k')) = U(k').$$

This means that, in our candidate equilibrium, firms are indifferent between offering $U(k)$ or strictly increasing the utility offered to both types to $U(k')$. But then supermodularity implies that increasing only the utility offered to the low-risk agent leads to a loss:

$$\begin{aligned} \pi(U_l(k'), U_h(k)) - \pi(U_l(k), U_h(k)) &= \int_{U_l(k)}^{U_l(k')} M(U_l(s), U_h(k)) U_l'(s) ds \\ &< \int_{U_l(k)}^{U_l(k')} M(U_l(s), U_h(s)) ds = 0. \end{aligned}$$

Offers that deliver to a type $t \in \{l, h\}$ utility u_t outside of support $[U_t(0), U_t(\bar{k})]$ can easily be ruled out: an offer that generates utility $u_l < U_l(0)$ never attracts low-risk types,

¹²Notice that $U_l'(0) > 0$ if $\bar{k} > 0$, $-\frac{\partial P_l(u(0))}{\partial u_l} > 0$, and both $U_l'(\cdot)$ and $\frac{\partial P_l(u(\cdot))}{\partial u_l}$ are continuous.

while making profits from high-risk types is impossible. A potential offer that generates utility $u_l > U_l(\bar{k})$ is dominated by another offer that generates exactly utility $U_l(\bar{k})$ to low-risk types: it still attracts low risks with probability 1 and makes higher profits in the case of acceptance.

In the [Appendix](#) we provide the complete proof that offering cross-subsidizing contracts according to distribution F is indeed an equilibrium. The proof uses the supermodularity property of the profit function in a similar way to show that all possible deviations are unprofitable.

PROPOSITION 1. *There exists a symmetric equilibrium such that (i) every firm randomizes over offers in $\{\mathcal{M}^k \mid k \in R\}$, where $k \in [0, \bar{k}]$ is distributed according to $F(\cdot)$, and (ii) after observing menu offers $(\mathcal{M}^{k_1}, \mathcal{M}^{k_2})$ with $k_i > k_j$, the agent chooses according to*

$$\begin{aligned} s(h, (\mathcal{M}^{k_1}, \mathcal{M}^{k_2})) &= (1 - p_h + k_i, 1 - p_h + k_i), \\ s(l, (\mathcal{M}^{k_1}, \mathcal{M}^{k_2})) &= \gamma(k_i). \end{aligned}$$

*After observing offers that are not of the form $(\mathcal{M}^{k_1}, \mathcal{M}^{k_2})$, the agent chooses any arbitrary selection from his best response set.*¹³

4.3 Pure strategy equilibrium

The equilibrium described in [Proposition 1](#) coincides with the pure strategy equilibrium characterized in [Rothschild and Stiglitz \(1976\)](#), whenever it exists. The equilibrium involves no mixing if and only if

$$\bar{k} = 0,$$

in which case the set of cross-subsidization levels offered in equilibrium is $R = \{0\}$ and firms offer the zero cross-subsidization menu $\mathcal{M}^0$, which coincides with the contract pair $\{c_l^{\text{RS}}, c_h^{\text{RS}}\}$.

[Lemma 2](#) shows that the benefits from cross-subsidization, from the low-risk agent's point of view, are quasi-concave. As a consequence zero cross-subsidization is Pareto optimal if and only if

$$U'_l(0) \leq 0. \tag{4}$$

The uniqueness result (discussed in [Section 5](#)) implies that (4) is a necessary and sufficient condition for the existence of a pure strategy equilibrium. The exact condition in terms of the prior distribution is presented in the following corollary.¹⁴

¹³With the restriction that s is still a mixed strategy, as defined in [Section 3](#).

¹⁴The conditions for the existence of pure strategy equilibria would be different if each firm is restricted to offering a single contract. In the menu game discussed here, firms can design more profitable deviations involving cross-subsidies; hence, the conditions under which a pure equilibrium exists are more demanding. It is worth noting that the characterization result for mixed equilibria used in this model does not extend to the single-contract game mentioned. The crucial feature of the analysis presented in this paper is the supermodularity of the mapping between the utility profile offered to different types and expected profits obtained by each firm. This property does not extend to the model where each firm offers a single contract. I thank one referee for pointing this out.

COROLLARY 1. *The equilibrium described involves pure strategies if and only if*

$$u'(c_h^{\text{RS}}) \left[\frac{1}{u'(c_{l,1}^{\text{RS}})} - \frac{1}{u'(c_{l,0}^{\text{RS}})} \right] \leq \frac{\mu_h}{\mu_l} \left[\frac{p_h}{p_l} - \frac{1-p_h}{1-p_l} \right].$$

If this is the case, all firms offer the pair of contracts $\{c_l^{\text{RS}}, c_h^{\text{RS}}\}$ in equilibrium.

A pure strategy equilibrium fails to exist whenever the share of low-risk agents is sufficiently high. In this case, the cost of cross-subsidization is very low since there are few high-risk agents to be subsidized.

4.4 The case $N > 2$

In the analysis of the duopoly case, I have shown that one can find a distribution over the set of menu offers $\mathbf{M}^R$ such that each firm finds it optimal to make any offer in this set.

This support $\mathbf{M}^R$ has the following property: the utility obtained by both types, $U_l(\cdot)$ and $U_h(\cdot)$, is strictly increasing in the cross-subsidization level k for $k \in R$. This means that if firm $i = 1$ faced two firms, 2 and 3, that were choosing offers $\mathcal{M}^k$ according to continuous distributions F_2 and F_3 , the relevant random variable for firm 1 would be $k_{23} = \max\{k_2, k_3\}$, which determines the only relevant threat from the offers of firms 2 and 3. The distribution of this variable is given by $F(k) = F_1(k)F_2(k)$. This allows us to adapt the arguments above by equalizing the distribution of the best among $N - 1$ firms with the single firm distribution in the duopoly.

PROPOSITION 2. *In the game with N firms, the following strategy profile represents an equilibrium: every firm randomizes over offer set $\mathbf{M}^R$ with distribution over cross-subsidization level $k \in R$ given by $F_i(\cdot)$, where*

$$F_i(k) = F(k)^{\frac{1}{N-1}}.$$

The equilibrium described has the following properties. First, whenever there is randomization, ties occur with zero probability: there is always a firm that offers M^{k_i} such that $k_i > \max_{j \neq i} k_j$. The agent gets a contract from this firm, independent of which type is realized. If the type is h , the agent ends up with contract $(1 - p_h + k_i, 1 - p_h + k_i)$. If the agent is of type l , he chooses contract $\gamma(k_i)$. Second, whenever the pure equilibrium with the RS contracts exists, $R = \{1 - p_h\}$ and the support of strategies reduces to the RS contracts.

In the next section, I show that this is the unique symmetric equilibrium. [Section 6](#) presents monotone comparative statics results regarding the prior distribution and the number of firms.

5. UNIQUENESS

In this section, we show that the equilibrium constructed in [Section 4](#) is the unique symmetric equilibrium. We start by showing that equilibrium offers can be fully described

by the utility they generate to both possible risk types. This means that describing the equilibrium strategies used by firms reduces to describing the equilibrium distribution of utility levels generated by equilibrium offers.

Describing offers in terms of utility profiles means that the offer space is essentially two dimensional. The main challenge in the analysis lies in showing that equilibrium offers necessarily lie in a one-dimensional subset of the feasible utility space. The crucial step uses properties of the equilibrium utility distribution and the ex post profit functions to show that expected profits are supermodular in the utility vector offered to the consumer. There is a complementarity in making more attractive offers to both risk types. Supermodularity is used to show that equilibrium offers are necessarily ordered in terms of attractiveness, i.e., a more attractive offer provides higher utility for both risk types.

Firms in this market make zero expected profits (as shown in [Proposition 3](#)). The ordering of offers is used to show that equilibrium offers necessarily generate zero profits even if they are accepted by both risk types with probability 1.¹⁵ Hence, offers can be indexed by the amount of subsidization that occurs across different risk types. The remaining analysis follows standard steps in the literature on games with one-dimensional strategy spaces (see [Lizzeri and Persico 2000](#) and [Maskin and Riley 2003](#)).

For an arbitrary mixed strategy ϕ for insurance firms, we denote as G the distribution of the highest utility for each type $t \in \{l, h\}$ induced by $N - 1$ offers generated according to distribution ϕ . Formally, define the utility obtained from offer $\mathcal{M} \in \mathbf{M}$ by an agent of type $t \in \{l, h\}$ as

$$u_t^{\mathcal{M}} \equiv \max\{U(c \mid t) \mid c \in \mathcal{M}\},$$

as for any $\mathbf{u} \in \mathbb{R}^2$,

$$G(u_l, u_h) \equiv [\phi\{\mathcal{M} \in \mathbf{M} \mid u_t^{\mathcal{M}} \leq u_t, \text{ for } t \in \{l, h\}\}]^{N-1}.$$

In equilibrium, G is relevant because it determines the distribution of outside options that any given firm faces when trying to attract a consumer. Also, we define as G_t the marginal of G over u_t for $t \in \{l, h\}$. The equilibrium outcome distribution is denoted as $\mathbb{P}^*$.

PROPOSITION 3 (Zero profits). *In any symmetric equilibrium, the following statements hold:*

- (i) *Firms make zero expected profits.*
- (ii) *Fix $t \in \{l, h\}$. If a firm i makes an offer $\mathcal{M}$ that generates utility profile $\mathbf{u}$, then $\chi_t(\mathbf{u}) \in \mathcal{M}$ and this is the only possible offer accepted by type t from firm i : for any $c \in \mathcal{M} \setminus \{\chi_t(\mathbf{u})\}$,*

$$\mathbb{P}^*[(\tilde{t}, s(\tilde{t}, (\mathcal{M}, \tilde{\mathcal{M}}_{-i}))) = (t, c, i)] = 0.$$

¹⁵Or if they are accepted in the market as a whole.

(iii) *Firms make nonnegative profits from low-risk agents and nonpositive profits from high-risk agents: any equilibrium utility offer $\mathbf{u}$ satisfies $\mathbf{u} \in \text{int}(Y)$, $u_l \geq u_h$, and*

$$P_l(\mathbf{u}) \geq 0,$$

$$P_h(\mathbf{u}) \leq 0.$$

The proof is provided in the [Appendix](#).

The previous statement contains three main results. First, it shows that firms never make positive expected profits in equilibrium. This follows from the assumption of Bertrand competition without differentiation. The main challenge in the proof is to deal with the multidimensionality of the offer space. Second, it shows that equilibrium offers can be described in terms of the utility profile they generate. If an offer $\mathcal{M}$ generates utility profile $\mathbf{u}$, then the only contracts from this set that can be consumed in equilibrium are $\{\chi_l(\mathbf{u}), \chi_h(\mathbf{u})\} \subseteq \mathcal{M}$. This comes from the fact that a firm's maximization problem can be split into choosing the attractiveness of the contract (given by the utility levels) and the minimization of costs for a fixed level of utility.¹⁶ Finally, a consequence of zero profits is that firms necessarily take (potentially zero) losses on high-risk agents and make profits on low-risk agents. The reason is that if an offer generates strictly positive profits from high-risk agents, a firm can always guarantee ex ante expected profits since low-risk agents are always at least as profitable as high-risk agents.

Given [Proposition 3](#), our uniqueness proof consists of showing that there exists only one possible equilibrium utility distribution generated by each firm. From part (ii) of the proposition, describing the distribution over utility profiles provides a description of equilibrium offers, namely $\chi(\mathbf{u})$.¹⁷

The following intermediary lemma provides a characterization of the equilibrium utility distribution G . It shows that this distribution is absolutely continuous except at one point: the utility level generated by Rothschild–Stiglitz offers. For each $t = l, h$, define $\underline{u}_t \equiv \inf\{u \mid G_t(u) > 0\}$ as the lowest equilibrium utility and, similarly, define $\bar{u}_t \equiv \sup\{u \mid G_t(u) < 1\}$ as the highest equilibrium utility.

LEMMA 6 (Utility distribution). *In any symmetric equilibrium, the utility distribution G satisfies the following conditions:*

- (i) *Lower bound on utility: $\underline{u}_t \geq u_t^{\text{RS}}$.*
- (ii) *Mass points for low risk: G_l has support $[\underline{u}_l, \bar{u}_l]$ and is absolutely continuous on $[\underline{u}_l, \bar{u}_l] \setminus \{u_l^{\text{RS}}\}$.*
- (iii) *Mass points for high risk: the only possible mass point of G_h is u_h^{RS} .*

¹⁶One has to consider the possibility of the agent being indifferent between two contracts. However, this is solved in the proof of [Proposition 3](#) by showing that a firm can always break such indifferences at infinitesimal cost.

¹⁷Obviously any equilibrium offer that generates utility $\mathbf{u}$ can include other contracts beyond $\{\chi_l(\mathbf{u}), \chi_h(\mathbf{u})\}$, as long as they are unattractive. This means they satisfy $U(c \mid t) \leq U(\chi_t(\mathbf{u}) \mid t)$. [Proposition 3](#) shows that if such contracts are present, they are irrelevant, meaning they are never chosen on the equilibrium path.

See the [Appendix](#) for the proof.

From now on, we denote the density of G_l , whenever it exists, as g_l .

[Proposition 3](#) shows that offers can be described in terms of the utility generated by them. [Lemma 6](#) shows that mass points are not possible, except at the utility generated from Rothschild–Stiglitz offers. This means that if a firm makes a utility offer $\mathbf{u} \gg (u_l^{\text{RS}}, u_h^{\text{RS}})$ in equilibrium, then its profits are given by

$$\pi(\mathbf{u}) \equiv \mu_l G_l(u_l) P_l(u) + \mu_h G_h(u_h) P_h(u).$$

In what follows, we show that the function $\pi(\cdot)$ is supermodular in (u_l, u_h) . This means that it satisfies an increasing differences property: the profit gains from making a more attractive offer to low-risk agents is strictly increasing with the utility offered to high-risk agents.

For such a utility offer to be optimal, there must be no alternative utility $\mathbf{u}'$ that increases expected profits. Consider utility offer $\mathbf{u}$ satisfying $u_l > u_h$ and $u_l > u_l^{\text{RS}}$. The marginal gain from making a more attractive offer to low-risk agents is given by¹⁸

$$M(\mathbf{u}) \equiv \mu_l \left[g_l(u_l) P_l(u) + G_l(u_l) \frac{\partial P_l(u)}{\partial u_l} \right],$$

where the first term captures the probability gain, which means that a more attractive offer has a higher chance of attracting low-risk individuals who generate positive profits. The second term captures the loss in profits that a firm faces so as to make a more attractive offer to low-risk agents. Using simple properties of the ex post profit function $P_l(\cdot)$ described in [Lemma 3](#), we show that the profit function satisfies supermodularity: the marginal profits from attracting low-risk agents is increasing in the utility offered to the high-risk agents. The following result differs from [Lemma 5](#) in that it deals with an arbitrary equilibrium distribution.

LEMMA 7 (Supermodularity). *For any feasible $\mathbf{u}$ satisfying $u_l > \max\{u_l^{\text{RS}}, u_h\}$, the function $M(u_l, \cdot)$ is nondecreasing in u_h . Moreover, if $u_l \in (\underline{u}_l, \bar{u}_l)$, then it is strictly increasing.*

The proof follows directly from the definition of $M(\cdot)$ and properties (iii) and (iv) in [Lemma 3](#).

Supermodularity implies that there is a complementarity between how attractive an offer is to low-risk and high-risk agents. This complementarity has an important implication for equilibrium offers: more attractive offers to low-risk agents have to be more attractive to high-risk agents also. In other words, equilibrium offers can be ordered in terms of attractiveness. This is formally stated in the next result.

¹⁸Notice that $u_l > u_h$ implies that

$$\frac{\partial P_h(\mathbf{u})}{\partial u_l} = 0.$$

This is true because the cost minimizing contract to be offered to the high-risk agent, $\chi_h(\mathbf{u})$, is efficient. This means that $\chi_h(\mathbf{u})$ is equal to the full insurance contract $\bar{c} = (u^{-1}(u_h), u^{-1}(u_h))$.

LEMMA 8 (Ordering of offers). *In any symmetric equilibrium, if the support of the equilibrium strategy includes an offer that generates utility profile $\mathbf{u} = (u_l, u_h)$, then*

$$u_h = v(u_l),$$

where the function $v : [\underline{u}_l, \bar{u}_l] \rightarrow \mathbb{R}$ is strictly increasing.

PROOF. Step 1. We first show that all equilibrium offers generating utility $\mathbf{u} = (u_l^{\text{RS}}, u_h^{\text{RS}})$ satisfy $u_h = u_h^{\text{RS}}$, i.e.,

$$G(u_l^{\text{RS}}, u_h^{\text{RS}}) = G_l(u_l^{\text{RS}}).$$

Suppose by way of contradiction that

$$G(u_l^{\text{RS}}, u_h^{\text{RS}}) < G_l(u_l^{\text{RS}}).$$

Hence, a given firm i , with positive probability, namely $q > 0$, makes offers that generate utility in the set $\{u_l^{\text{RS}}\} \times (u_h^{\text{RS}}, \infty)$. For an arbitrary offer $\mathbf{u} = (u_l^{\text{RS}}, u_h)$ in this set, we know that

$$P_h(u_h) < 0.$$

So it is necessarily the case that the firm makes positive profits from low-risk agents, which means that

$$P_l(\mathbf{u}) > 0.$$

Since, at offer $\mathbf{u}$, the firm makes positive profits from the low-risk agent, it must attract the low-risk agent with probability $G_l(u_l^{\text{RS}})$.¹⁹ But this leads to a contradiction: if firm $j \neq i$ makes offers in $\{u_l^{\text{RS}}\} \times (u_h^{\text{RS}}, \infty)$, then it attracts the low-risk agent with probability at most

$$G_l(u_l^{\text{RS}})^{\frac{N-2}{N-1}} (G_l(u_l^{\text{RS}})^{\frac{1}{N-1}} - q) < G_l(u_l^{\text{RS}}),$$

since it cannot attract the low-risk agent if firm i makes an offer in the set $\{u_l^{\text{RS}}\} \times (u_h^{\text{RS}}, \infty)$. Hence, offer $\mathbf{u}$ cannot be optimal to firm j , a contradiction.

We now consider $u_l > u_l^{\text{RS}}$.

Step 2. There is a unique $u_h \in \mathbb{R}$ such that an offer generating utility vector $\mathbf{u} \equiv (u_l, u_h)$ is offered. If an offer generating utility $\mathbf{u}$ is made, then optimality of $\mathbf{u}$ implies that

$$M(\mathbf{u}) = 0.$$

However, since $M(u_l, \cdot)$ is strictly increasing, by Lemma 7, there is a unique u_h that satisfies this equality. We call this utility level $v(u_l)$.

Step 3. The function $v(\cdot)$ is nondecreasing for $u_l > u_l^{\text{RS}}$. Suppose that both utility vectors $(u_l, v(u_l))$ and $(u'_l, v(u'_l))$ are offered in equilibrium with $u'_l > u_l$. Optimality implies that

$$\pi(u'_l, v(u_l)) - \pi(u_l, v(u_l)) \leq 0 \leq \pi(u'_l, v(u'_l)) - \pi(u_l, v(u'_l));$$

¹⁹If not, then offer $\mathbf{u}' = \mathbf{u} + (\varepsilon, 0)$ is strictly better for $\varepsilon > 0$ small since it attracts the low-risk agent with at least probability $G_l(U(c^{\text{RS}, l} | l))$ while incurring an arbitrarily small extra cost (P_h is continuous).

however, this implies that

$$\int_{u_l}^{u'_l} M(s, v(u_l)) ds \leq 0 \leq \int_{u_l}^{u'_l} M(s, v(u'_l)) ds,$$

which implies that $v(u'_l) \geq v(u_l)$.

Step 4. Consider $v(u_l) > u_h^{\text{RS}}$ for $u_l > u_l^{\text{RS}}$. Suppose that an offer generating utility $\mathbf{u} = (u_l, u_h)$ with $u_l > \max\{u_l^{\text{RS}}, u_h\}$ and $u_h = u_h^{\text{RS}}$. By definition, we have that

$$P_t(u_t^{\text{RS}}) = 0, \quad \text{for } t \in \{l, h\}.$$

Notice that $P_h(\mathbf{u}) \leq 0$ since the high risk is receiving the utility generated by his actuarially fair full insurance policy. Also notice that, since $P_l(\cdot, u_h)$ is strictly decreasing, $P_l(\mathbf{u}) < 0$. Hence, this means that the offering firm makes strictly negative profits: low-risk agents are attracted by this offer with probability $G_l(u_l) > 0$.

Step 5. The function $v(\cdot)$ is strictly increasing for $u_l > U(c^{\text{RS}, l} | l)$. Suppose that $u_l, u'_l \in (\underline{u}_l, \bar{u}_l)$ and that $v(u_l) = v(u'_l) > u_l^{\text{RS}}$. Then we know, from Step 1, that $v(s) = v(u_l)$ for any $s \in (u_l, u'_l)$. Hence it follows that

$$G_h(v(u_l)) \geq G_l(u'_l) - G_l(u_l) > 0.$$

But if $G_l(\underline{u}_l) > 0$, then [Lemma 6](#) implies $\underline{u}_l = U(c^{\text{RS}, l} | l)$. This contradicts Step 1. $\square$

This result reduces the set of equilibrium utilities to a one-dimensional subset of Y , since all offers can be indexed by the low-risk agent's utility level. The zero profits result in [Proposition 3](#) implies that offers can be indexed as well by the amount of cross-subsidization across different types.

COROLLARY 2. *In any symmetric equilibrium, all equilibrium offers generate a utility profile in the set*

$$\{U(k) \mid k \in R\}.$$

PROOF. From [Proposition 3](#), we know that firms make zero expected profits. Now if a firm makes offer $\mathbf{u} = (u_l, v(u_l))$ with $u_l > \underline{u}_l$, [Lemma 8](#) implies that

$$G_h(v(u_l)) = G_l(u_l).$$

Then profits are given by

$$\pi(\mathbf{u}) \equiv G_l(u_l)[\mu_l P_l(\mathbf{u}) + \mu_h P_h(\mathbf{u})] = 0.$$

Function $P_l(\cdot, u_h)$ is strictly decreasing and $P_l(z, u_h) \rightarrow -\infty$ as $z \rightarrow \lim_{c \rightarrow \infty} u(c)$. So any equilibrium offer utility $\mathbf{u} = (u_l, u_h)$ satisfies (i) $u_h \in [u_h^{\text{RS}}, U_h(p_h - \bar{p})]$ and (ii) $\mathbf{u} = U(k)$ for $k = U_h^{-1}(u_h)$. To check (i), suppose that $U_h > u(p_h - \bar{p})$. Then we have a contradiction since

$$\mu_l P_l(\mathbf{u}) + \mu_h P_h(\mathbf{u}) \leq \mu_h P_h(u_h, u_h) + \mu_l P_l(u_h, u_h) < 0.$$

To check (ii), notice that, for any $k \in (0, p_h - \bar{p})$, $U_l(k)$ maximizes the utility obtained by the low-risk subject to delivering utility $U_h(k)$ to the high-risk agent and generating zero expected profits. Hence $U_l(k)$ is the unique solution to

$$\mu_h P_h(u_l, U_h(k)) + \mu_l P_l(u_l, U_h(k)) = 0.$$

Finally, suppose by way of contradiction that $\bar{u}_l > U_l(\bar{k})$. Then take equilibrium utility offers $\mathbf{u} = (U_l(k), U_h(k))$ and $\mathbf{u}' = (U_l(k'), U_h(k'))$ such that $U_l(\bar{k}) < U_l(k) < U_l(k') < \bar{u}_l$. Lemma 8 implies that

$$U_h(k) < U_h(k') \Rightarrow k' > k.$$

But the fact that $k' > k > \bar{k}$ implies that

$$U_h(k') < U_h(k) < U_h(\bar{k}),$$

since $u_h(\cdot)$ is concave and $\bar{k}$ is its peak, a contradiction. $\square$

The importance in the previous lemma is in reducing the set of possible utility profiles to a one-dimensional set, where offers are indexed by the amount of cross-subsidization occurring between different risk types. This allows us to use standard tools from games with one-dimensional strategy spaces to describe the unique possible equilibrium distribution over this feasible set.

PROPOSITION 4 (Uniqueness). *Consider any symmetric equilibrium. Then the set of utility profiles generated by equilibrium offers by any single firm is*

$$\{\mathbf{u}(k) \mid k \in R\},$$

where variable $k \in R$ is distributed according to $F^* = F^{\frac{1}{N-1}}$, where F is presented in Lemma 4.

PROOF. First we show that $\text{supp}(G) = \{U(k) \mid k \in R\}$. Suppose that $\bar{u}_l = U_l(k)$ such that $k < \bar{k}$. Then a firm can make utility offer $(U_l(\frac{k+\bar{k}}{2}) - \varepsilon, U_h(\frac{k+\bar{k}}{2}))$ for $\varepsilon > 0$ satisfying

$$U_l\left(\frac{k+\bar{k}}{2}\right) - \varepsilon > U_l(k).$$

This would attract both risk types with probability 1 and make positive expected profits.

Now suppose that $\underline{u}_l = U_l(\underline{k}) > U_l(0)$. Then one firm can make utility offer

$$(U_l(\underline{k}) + \varepsilon, U_h(\underline{k}) - \varepsilon)$$

with $\varepsilon > 0$ sufficiently small so that

$$P_l(U_l(\underline{k}) + \varepsilon, U_h(\underline{k}) - \varepsilon) > 0.$$

This would attract the high-risk agent with zero probability, attract the low-risk agent with positive probability, and make positive expected profits.

Finally, let the distribution over $k \in R$ be F^* and define $F = (F^*)^{N-1}$ as the distribution of the highest draw from $N - 1$ levels of cross-subsidization. Notice that the distribution of the best competing offer for a low-risk agent a firm faces satisfies $G_l(U_l(k)) = F(k)$.

Since any offer in $\{u(k) \mid k \in R\}$ must be optimal, the distribution over cross-subsidization $k \in R$ must satisfy

$$\frac{f(k)}{u'_l(k)} P_l(U(k)) + F(k) \frac{\partial P_l(U(k))}{\partial u_l} = 0,$$

and $F(\bar{k}) = 1$. But the unique solution to these conditions is given by [Lemma 4](#). $\square$

6. COMPARATIVE STATICS

Since our uniqueness result is valid for all possible number of firms and prior distributions over types, an analysis of the comparative statics with respect to these primitives of the model is possible and informative.

6.1 Prior distribution

One of the main advantages of considering an extensive form version of Rothschild and Stiglitz's model is to obtain equilibrium existence for all prior distributions. Here I consider how equilibrium strategies change with the prior probability of the low-risk agent. The equilibrium distribution of offers continuously changes with the prior distribution, converging to the RS contracts for μ_l sufficiently small and converging to the full information full insurance offer $(1 - p_l, 1 - p_l)$ as μ_l converges to 1. More specifically, this means that both agents benefit from a better pool of agents. This result conflicts with those obtained by [Bisin and Gottardi \(2006\)](#), [Dubey and Geanakoplos \(2002\)](#), and [Guerrieri et al. \(2010\)](#), who consider extensions of the original Rothschild–Stiglitz model.²⁰ In their papers, equilibria generate the RS pair of contracts as a final outcome for any prior distribution. I consider a fixed number N of firms and I define $F^{(\mu_l)}$ as the equilibrium distribution of offers by a single firm. Also, denote as $\bar{F}^{(\mu_l)}$ the equilibrium distribution of the best offers, i.e., the distribution of $k = \max\{k_1, \dots, k_N\}$.

PROPOSITION 5. *If $\mu'_l > \mu_l$, then $F^{(\mu'_l)}$ first-order stochastically dominates $F^{(\mu_l)}$ and $\bar{F}^{(\mu'_l)}$ first-order stochastically dominates $\bar{F}^{(\mu_l)}$. When $\mu_l \rightarrow 1$, $F^{(\mu_l)}$ and $\bar{F}^{(\mu_l)}$ converge to a point mass at p_l . Moreover, the function $\mu_l \mapsto F^{(\mu_l)}$ is continuous (weak convergence).*

6.2 Number of firms

In this section, we show that an increase in the number of competing firms leads to a decrease in the level of cross-subsidization in the market, which affects the welfare

²⁰As mentioned in the Introduction, continuity with respect to the distribution is present in the dynamic extensions considered in [Wilson \(1977\)](#), [Miyazaki \(1977\)](#), [Hellwig \(1987\)](#), [Mimra and Wambach \(2011\)](#), and [Netzer and Scheuer \(2014\)](#).

of all consumers negatively.²¹ This result is reminiscent of that presented in [Rosenthal and Weiss \(1984\)](#) for the Spence model and is at odds with the classical literature on oligopoly.²²

The contract distribution in our model is determined to be the deterrence of cream-skimming deviations. For each firm, the relevant market variable is the best alternative menu offer made by a competitor. Hence, the distribution of the best alternative offer among $N - 1$ draws is the equilibrium object pinned down by the absence of profitable cream-skimming deviations by a single firm and, hence, is independent of N . In other words, the equilibrium response to a larger number of firms is the use of less attractive offers by each firm in a way that the distribution of the best competing offer faced by any single firm remains unchanged. Consumers are more affected by changes in equilibrium strategies than firms, since they observe N different offers while each firm competes among $N - 1$ alternative offers, and as a consequence the use of less attractive offers by each firm leaves consumers worse off with an increase in the number of firms. For the remainder of this section, let $F^{(N)}$ denote the equilibrium distribution over cross-subsidization level $k \in R$ used by each firm.²³

PROPOSITION 6. *The distribution $F^{(N)}$ first-order stochastically dominates $F^{(N+1)}$. In the case $\bar{k} > 0$, the dominance is strict. Additionally, $F^{(N)}$ converges weakly to a Dirac measure at the zero cross-subsidization level as $N \rightarrow \infty$.*

PROOF. Just notice that, using the result in [Proposition 2](#),

$$F^{(N)} = F(k)^{\frac{1}{N-1}} \leq F(k)^{\frac{1}{N}} = F^{(N+1)}.$$

In the case $\bar{k} > 1 - p_h$, there is continuous mixing over $[0, \bar{k}]$, so that the inequality is strict for any $k \in (0, \bar{k})$.

Finally, if the distribution F is a point mass at zero, the convergence of $F^{(N)}$ is trivial. In case F is continuous on $[0, \bar{k}]$, notice that for any $k \in (0, \bar{k})$,

$$F(k) \in (0, 1) \quad \Rightarrow \quad F(k)^{\frac{1}{N-1}} \xrightarrow{N \rightarrow \infty} 1. \quad \square$$

²¹Since the number of firms is assumed to be exogenous in our model, it is the critical parameter related to the intensity of competition. An interesting extension of this game introduces a stage of simultaneous costly entry decision by firms followed by competition with a public number of firms. This game has a unique symmetric equilibrium in mixed strategies where the ex ante expected profits from entry, which occur if a firm ends up being a monopolist, are equal to the entry cost. In such a game, an increase in the entry cost leads to a decrease in the (random) number of active firms in terms of first-order stochastic dominance. However, the overall ex ante welfare effect is ambiguous since the welfare of consumers is a nonmonotonic function of the number of firms in the market: it is easy to show that the welfare of each risk type under monopoly ($N = 1$) is lower than under competition ($N > 1$) while [Proposition 6](#) shows that for $N > 2$, an increase in the number of firms decreases welfare in the market. I thank a referee for raising this subtle point.

²²A notable exception is [Janssen and Moraga-González \(2004\)](#), who consider a model with search frictions. Their model displays multiple equilibria, but welfare is decreasing in the number of firms when focusing on equilibria with low intensity of search.

²³The positive connection between number of firms and welfare, which is present in the seminal Cournot model of competition, is driven by the rent extraction motive of firms, which is not present in the current model as firms always make zero expected profits in equilibrium.

Since the utility provided by offer $\mathcal{M}^k$ is increasing in k , for both types, the first part of the proposition implies that the utility delivered by a single firm decreases as N increases. However, for a higher number of firms, the agent is sampling a higher number of offers, so that the overall effect seems unclear. But the distribution of the best offer among any $N - 1$ firms is independent of N ; hence, the distribution of the best offer among N firms is lower (in first-order stochastic dominance) and converges to $F^{(2)}$. Let $\bar{F}^{(N)}$ be the distribution of $\max_{i=1,\dots,N} k_i$, where each k_i is distributed according to $F^{(N)}$.

PROPOSITION 7. *The distribution $\bar{F}^{(N)}$ first-order stochastically dominates $\bar{F}^{(N+1)}$. Moreover,*

$$\bar{F}^{(N)} \rightarrow^w F^{(2)}$$

as $N \rightarrow \infty$.

To get the proof, just notice that $\bar{F}^{(N)} = F^{(2)}F^{(N)}$.

The consequence of the propositions above is that when the RS pure equilibrium fails to exist, the model with a continuum of firms cannot simply be taken as the limit of a model with N firms, as $N \rightarrow \infty$. The problem is that as the number of firms grow, each firm provides worse offers. However, their offers get worse “slowly” so that the best offer among N firms converges to a nondegenerate distribution F . In the case of a continuum of firms, there is no way to obtain a nondegenerate distribution for the best offer among all firms with independent and identical randomizations across firms.

An important characteristic of the mixed equilibrium constructed is that an outside firm, facing equilibrium offers in the market, can obtain positive expected profits. Consider the duopoly case. An outside firm (called firm 3) faces two competing offers (from firms 1 and 2) distributed according to F , so that the most attractive competing offer is distributed according to F^2 . If an outside firm considers any cross-subsidizing offer $\mathcal{M}^k$, it would have zero expected profits by the definition of $\mathcal{M}^k$. But since firms 1 and 2 have zero expected gains from the local deviation around $\mathcal{M}^k$, when facing competing offers distributed according to F , firm 3 has a strict gain from a small deviation that attracts low-risk agents with higher probability. It is surprising that firm 3, facing two competing offers distributed according to F , can obtain higher expected profits than a firm facing a single competing offer. In most competitive settings, such as in auctions, a player always benefits from less aggressive offers from its competitors. In this model, however, cross-subsidization between contracts means that the relative (as opposed to absolute) frequency with which an offer attracts both types determines profits.

7. CONCLUSION

In this paper, I consider a competitive insurance model in which a finite number of firms simultaneously offer menus of contracts to an agent with private information regarding his risk type. I show that there always exists a unique symmetric equilibrium. This equilibrium features firms offering the separating contracts analyzed in [Rothschild and Stiglitz \(1976\)](#) whenever they can be sustained as an equilibrium outcome. When this

is not the case, which occurs if the prior probability of low-risk agents is too high, firms randomize over a set of separating pairs of offers. Firms obtain zero expected profits in equilibrium.

The equilibrium features monotone comparative statics with respect to the prior over types and the number of firms. As the probability of low-risk agents increases, firms offer more attractive menus. The equilibrium is continuous with respect to the prior. As a consequence, equilibrium outcomes converge to the perfect information allocation when the prior converges to both extremes. Regarding the number of firms, the distribution of the best offer in the market and agent's welfare decrease as it grows. The distribution of the best offer converges to the mixed strategy of a single firm in duopoly.

Due to the implicit nature of the equilibrium construction, I am not able to obtain clear comparative statics results with respect to preferences. Numerical exercises suggest that higher constant risk aversion leads to more aggressive offers by the firms and reduces the set of priors for which a pure equilibrium exists.

An interesting issue is the extension of this equilibrium for an arbitrary number of types, since the equilibrium existence problem presented by Rothschild and Stiglitz becomes more severe as the number of types increases. For the limiting case of a continuum of types, a competitive equilibrium never exists (see [Riley 1979, 2001](#)). In the two types model considered here, every optimal pair of contracts has to satisfy one local optimality condition, which is used to characterize the equilibrium distribution. If there are n potential types, there are $n - 1$ such local conditions. All of these conditions have to be simultaneously satisfied at any n -tuple offered in equilibrium. In the case of two types, the region of offers is given by γ and corresponds to the pairs of separating contracts that generate expected zero profits, and is a one-dimensional object. In the case of n risk types, it is an $n - 1$ -dimensional object, namely tuples, that provide full insurance to the lowest type, leave any given type indifferent between his allocation and the next higher type, and generate zero expected profits. The extra $n - 2$ local optimality conditions characterize the one-dimensional region in which the randomization occurs. The local condition connected to the highest type characterizes the equilibrium distribution. Given the complexity and relevance of the binary type analysis, this paper restricts attention to this case.

The analysis presented here sheds new light on the classical results on competitive insurance such as nonexistence of equilibrium, uniqueness, and the welfare impact of private information. However, there are issues with the interpretation of equilibrium in the insurance market when it involves mixed strategies. The outcome described here requires uncertainty with respect to the competing offers each firm faces. In actual insurance markets, this uncertainty can be generated by unobserved firm heterogeneity or from a combination of price dispersion and limited search by consumers

APPENDIX

A.1 *Auxiliary lemmas*

In this section, we present auxiliary notation that is used extensively in the proofs and a characterization of the feasible set of utility profiles that can be generated by incentive compatible contracts.

Denote $s_i(t, (\mathcal{M}_i)_i) \in \Delta \mathbb{R}_+^2$ as the the probability of acceptance of firm i 's contracts, i.e., for any measurable set $A \subseteq \mathbb{R}_+^2$, it is defined as

$$s_i(A | t, (\mathcal{M}_i)_i) = s(A \times \{i\} | t, (\mathcal{M}_i)_i).$$

Expected profit for firm i from offer $\mathcal{M}_i$ is

$$\begin{aligned} \pi_i(\mathcal{M}_i) &= \overbrace{\mu_h \int \left[\int \Pi(c | h) s_i(dc | h, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}) \right] d(\times_j \phi(\overline{\mathcal{M}}_{-i}))}^{\equiv \pi_{i,h}(\mathcal{M}_i)} \\ &+ \underbrace{\mu_l \int \left[\int \Pi(c | l) s_i(dc | l, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}) \right] d(\times_j \phi(\overline{\mathcal{M}}_{-i}))}_{\equiv \pi_{i,l}(\mathcal{M}_i)}. \end{aligned}$$

Also denote the ex ante expected probability of acceptance of an offer from firm i if the consumer is of type $t = l, h$ as $s_{i,t}(\mathcal{M})$ and denote the ex ante average profit made from a type $t = l, h$ agent conditional on a contract from firm i being accepted as $\pi_{i,t}^E(\mathcal{M})$. Define

$$u_t^{\mathcal{M}} \equiv \sup_{c \in \mathcal{M}} U(c | t).$$

Consequences of equilibrium conditions are the following. The acceptance rule always has to satisfy

$$c \in \text{supp}(s_i(t, (\mathcal{M}_i)_i)) \Rightarrow c \in \arg \max_{c \in \bigcup_i \mathcal{M}_i' \cup \{Y\}} U(c | t),$$

which implies that

$$c \in \text{supp}(s_i(t, (\mathcal{M}_i)_i)) \Rightarrow \Pi(c | t) \leq P_t(u_l^{\mathcal{M}}, u_h^{\mathcal{M}})$$

and

$$G_t^-(u_t^{\mathcal{M}}) \leq s_{i,t}(\mathcal{M}) \leq G_t(u_t^{\mathcal{M}}),$$

where $G_t^-(u) \equiv \lim_{k \nearrow u} G(k)$ is the left limit of distribution G_t .

Firms maximize profits, i.e.,

$$\pi_i(\mathcal{M}_i) \geq \pi_i(\mathcal{M}_i')$$

for all $\mathcal{M}_i \in \text{supp}(\phi)$ and $\mathcal{M}_i' \in \mathbf{M}$.

The set of feasible utility profiles $\mathbf{Y}$ has the following properties, which follow from standard convexity arguments:

- (i) $\mathbf{Y}$ is convex
- (ii) $(u(c), u(c)) \in \mathbf{Y}$, for all $c \in \mathbb{R}_{++}$
- (iii) $\mathbf{u} \in \mathbf{Y} \Rightarrow u + (\varepsilon, \varepsilon) \in \text{int}(\mathbf{Y})$ for $\varepsilon > 0$ sufficiently small.

The following lemma is also important in the proof and eliminates the possibility of equilibrium offers that are in the frontier of the set Y .

LEMMA 9. *For any $\mathbf{u} \in Y$ such that $P_h(\mathbf{u}) \leq 0$ and $P_l(\mathbf{u}) \geq 0$, then $\mathbf{u} + (\varepsilon, -\varepsilon) \in \text{int}(Y)$ for $\varepsilon > 0$ sufficiently small.*

PROOF. First, suppose that $u_h > u_l$. In this case the set²⁴

$$\text{co}\{\mathbf{u}, (u(0), u(0)), (u_h, u_h)\}$$

is contained in Y since the three generating elements are in Y . Finally notice that $\mathbf{u} + (\varepsilon, -\varepsilon)$ is in the interior of this set for $\varepsilon > 0$ sufficiently small.

Second, if $u_h = u_l$, then $\mathbf{u}$ is clearly in the interior of Y .

Third, now consider $u_h < u_l$. Condition $P_h(\mathbf{u}) \leq 0$ implies that $u_h > u(1 - p_h) = u_h^{\text{RS}}$. Define

$$C(\mathbf{u}) \equiv \{(c_l, c_h) \mid U(c_h \mid l) \leq U(c_l \mid l) = u_l; U(c_l \mid h) \leq U(c_h \mid h) = u_h\}.$$

If there exists $(c_l, c_h) \in C(\mathbf{u})$ such that $c_l \in \mathbb{R}_{++}^2$, then $\mathbf{u} + (0, \varepsilon) \in Y$ since

$$\left(c_h, c_l + \left(\frac{p_h}{p_h - p_l}\varepsilon, -\frac{(1 - p_h)}{p_h - p_l}\varepsilon\right)\right) \in C(\mathbf{u} + (0, \varepsilon)).$$

In this case, we know that $\mathbf{u} \in \text{int}(Y)$ since

$$\mathbf{u} \in \text{int}\{\text{co}\{\mathbf{u} + (0, \varepsilon), (u(0), u(0)), \mathbf{u} + (\varepsilon, \varepsilon)\}\}.$$

But if $(c_l, c_h) \in C(\mathbf{u})$ implies that $c_l = (0, k)$ for some $k > 0$, then it is necessarily the case that

$$(1 - p_h)u(k) = u_h \geq u_h^{\text{RS}}.$$

But then $k = u^{-1}(\frac{u_h}{1 - p_h}) \geq k^{\text{RS}} \equiv u^{-1}(\frac{u_h^{\text{RS}}}{1 - p_h})$. But allocation $(0, k^{\text{RS}})$ generates utility u_h^{RS} to the high-risk agent while introducing the maximal amount of risk, which implies that it generates utility strictly above u_l^{RS} to the low-risk agent. Since $(0, k^{\text{RS}})$ generates utility above $U(c_l^{\text{RS}} \mid l) = u_l^{\text{RS}}$ and has higher risk than c_l^{RS} , it follows that

$$\Pi((0, k^{\text{RS}}) \mid l) \leq \Pi(c_l^{\text{RS}} \mid l) = 0.$$

Hence the allocation $(0, k)$ necessarily generates negative losses as it pays more than $(0, k^{\text{RS}})$. $\square$

And last, the following lemma shows that weak incentive constraints can be turned into strict inequalities with arbitrarily small cost.

²⁴For any set $A \subseteq \mathbb{R}^2$, we use the notation $\text{co}(A)$ to denote the convex hull of set A .

LEMMA 10 (Costless strict incentives). *Consider $\mathbf{u} \in \text{int}(\mathbf{Y})$. For any $\varepsilon > 0$, there exists a pair of contracts $(c_l^\varepsilon, c_h^\varepsilon)$ such that*

$$\begin{aligned} U(c_l^\varepsilon | l) &= u_l > U(c_h^\varepsilon | l), \\ U(c_h^\varepsilon | h) &= u_h > U(c_l^\varepsilon | h), \\ \|\Pi(c_t^\varepsilon | t) - P_t(\mathbf{u})\| &\leq \varepsilon \quad \text{for } t = l, h. \end{aligned}$$

PROOF. Fix $\varepsilon > 0$. Since $\mathbf{u}$ is in the interior of $\mathbf{Y}$ and $P_t(\cdot)$ is convex (and hence continuous on the interior of its finite domain) for $\gamma > 0$ sufficiently small,

$$\begin{aligned} \|P_h(u_l - \delta, u_h) - P_h(u_l, u_h)\| &< \varepsilon, \\ \|P_l(u_l, u_h - \delta) - P_l(u_l, u_h)\| &< \varepsilon. \end{aligned}$$

Then the pair $(\chi_l(u_l, u_h - \delta), \chi_h(u_l - \delta, u_h))$ satisfies all the inequalities above. $\square$

A.2 Proof of Lemma 3

First notice that if $\mathbf{u}$ satisfies $u_l \leq u_h$, the optimal allocation is $\chi_l(\mathbf{u}) = (z, z)$, where $u(z) = u_l$.

If $\mathbf{u}$ satisfies $u_l > u_h$, then both constraints bind in the optimal allocation and, hence,

$$\chi_l(\mathbf{u}) = \left(u^{-1} \left(\frac{(1 - p_l)u_h - u_l(1 - p_h)}{p_h - p_l} \right), u^{-1} \left(\frac{p_h u_l - p_l u_h}{p_h - p_l} \right) \right).$$

Hence, $\chi_l(\cdot)$ is continuous, continuously differentiable if $u_l \neq u_h$, and satisfies $\chi_{l,1}(\mathbf{u}) \geq \chi_{l,0}(\mathbf{u})$ with strict inequality if $u_l > u_h$. Function $P_l(\cdot)$ is equal to

$$P_l(\mathbf{u}) = 1 - p_l - [p_l \chi_{l,0}(\mathbf{u}) + (1 - p_l) \chi_{l,1}(\mathbf{u})]$$

and, hence, it is continuous and continuously differentiable in $\mathbf{Y}_+ \equiv \{\mathbf{u} \in \mathbf{Y} \mid u_l > u_h\}$ and $\mathbf{Y}_- \equiv \{\mathbf{u} \in \mathbf{Y} \mid u_l < u_h\}$.

Direct differentiation leads to

$$\frac{dP_l(\mathbf{u})}{d(u_l, u_h)} = \begin{bmatrix} -\frac{1}{u'(u^{-1}(u_l))} \\ 0 \end{bmatrix} \quad \text{if } \mathbf{u} \in \mathbf{Y}_-$$

and

$$\frac{dP_l(\mathbf{u})}{d(u_l, u_h)} = \begin{bmatrix} -\frac{1}{p_h - p_l} \left[\frac{(1 - p_l)p_h}{u'(\chi_{l,1}(\mathbf{u}))} - \frac{p_l(1 - p_h)}{u'(\chi_{l,0}(\mathbf{u}))} \right] \\ \frac{p_l(1 - p_l)}{p_h - p_l} \left[\frac{1}{u'(\chi_{l,1}(\mathbf{u}))} - \frac{1}{u'(\chi_{l,0}(\mathbf{u}))} \right] \end{bmatrix} \quad \text{if } \mathbf{u} \in \mathbf{Y}_+.$$

Notice that $\frac{dP_l}{d\mathbf{u}}(\mathbf{u})$ is continuous at any point $\mathbf{u}$ with $u_l = u_h$ and, hence, P_l is continuously differentiable.

Since $u'(\cdot)$ is continuously differentiable, P_l is twice continuously differentiable in Y_+ and Y_- .

For any $\mathbf{u} \in Y_+$:

$$\frac{\partial^2 P_l}{\partial u_h \partial u_l} = -\frac{p_l(1-p_l)}{(p_h-p_l)^2} \left[\frac{(1-p_h)u''(\chi_{l,0})}{u'(\chi_{l,0})^2} + \frac{p_h u''(\chi_{l,1}(\mathbf{u}))}{u'(\chi_{l,1}(\mathbf{u}))^2} \right] > 0,$$

and for any $u \in Y_-$, the function $\frac{\partial P_l}{\partial u_h}$ is identically zero and, hence, is continuously differentiable.

A.3 Proof of Proposition 1

Since, in equilibrium, the distribution G of utility vectors generated by offers is continuous, any offer $\mathcal{M}$ is dominated by offer $\chi(\mathbf{u})$, where $\mathbf{u} = (u_l^{\mathcal{M}}, u_h^{\mathcal{M}})$. Offer $\chi(\mathbf{u})$ attracts the agent with the same probability as $\mathcal{M}$ and makes weakly more profits, conditional on attracting the agent. Henceforth, we focus on deviations of this form.

Case A: $u_l = U_l(k)$ and $u_h = U_h(k')$ for some $k, k' \in R$. Suppose that $k' > k$, which implies that

$$u_h = U_h(k') > U_h(k).$$

Hence, it follows from Lemma 5 that for any $\tilde{u} \in (U_l(k), U_l(k'))$,

$$M(\tilde{u}, u_h) > 0,$$

and, hence, offer $\chi(U_l(k'), U_h(k'))$ strictly dominates the original offer. An analogous proof follows if $k' < k$.

Case B: $u_l < U_l(0)$. Any offer that generates utility below $U_l(0)$ attracts a low-risk agent with zero probability. But there are no profit opportunities on high-risk agents since their equilibrium utility is above u_h^{RS} .

Case C: $u_l > U_l(0)$ and $u_h < U_h(0)$. By construction, the utility pair $(U_l(0), U_h(0))$ satisfies $P_l(U_l(0), U_h(0)) = 0$. Since P_l is decreasing in u_l and increasing in u_h , it follows that $P_l(\mathbf{u}) < 0$. Since there are no profit opportunities from high-risk agents, a firm cannot make positive profits by offering $\mathbf{u}$.

Case D: $u_l > U_l(\bar{k})$. Any such offer is dominated by an offer with $u_l = U_l(\bar{k})$, which offers strictly lower utility to the low-risk agent while still attracting the agent with probability 1.

Case E: $u_h > U_h(\bar{k})$ and $u_l = U_l(k)$ for some $k \in R$. If $\mathbf{u} \in \text{int}(Y)$, then Lemma 5 implies that

$$M(\mathbf{u}) > 0$$

and, hence, offer $\mathbf{u}$ is strictly dominated by offer $\mathbf{u} + (\varepsilon, 0)$ for $\varepsilon > 0$ small. If $\mathbf{u} + (\varepsilon, 0)$ is not feasible, Lemma 9 implies that $P_l(\mathbf{u}) < 0$. Hence, offer $\mathbf{u}$ cannot be profitable.

A.4 Proof of Proposition 3

I have separated the result into different three different lemmas.

LEMMA 11. (*Zero profits*). *In any symmetric equilibrium, firms make zero expected profits.*

PROOF. Suppose, by way of contradiction, that firm i makes expected profits $\pi_0 > 0$ and define

$$\underline{u}_i \equiv \inf\{u \in \mathbb{R} \mid G_i(u) > 0\}.$$

The proof is divided into four subcases.

Case 1. Suppose that $G_l(\underline{u}_l) = 0$. There exists a sequence of on-path contracts $\mathcal{M}_n$ such that $u_{l,n} \equiv u_l^{\mathcal{M}_n} \searrow \underline{u}_l$ and $G_l(\underline{u}_l) < \frac{1}{n}$. Let $u_{h,n} \equiv u_h^{\mathcal{M}_n}$ and $\mathbf{u}_n \equiv (u_{l,n}, u_{h,n})$.

Expected profits from low-risk agents, $\pi_{i,l}(\mathcal{M}_n)$, are at most $\frac{\mu_l}{n}(1 - p_l)$. It then follows that

$$\pi_0 \leq \frac{\mu_l}{n}(1 - p_l) + \pi_{i,h}(\mathcal{M}_n) \Rightarrow \pi_{i,h}(\mathcal{M}_n) \geq \pi_0 - \frac{\mu_l}{n}(1 - p_l),$$

and profits from high-risk agents are positive and bounded away from zero for n large. This means that $(1 - p_l)u(1) \leq u_{h,n} < u(1 - p_h) - k$ for some $k > 0$ and n large. Potentially passing to a subsequence, $u_{h,n}$ converges to $\tilde{u}_h \in [(1 - p_l)u(1), u(1 - p_h))$.

Also, for each n , the firm cannot deviate by offering $(u_{h,n} + \varepsilon, u_{h,n} + \varepsilon)$ for $\varepsilon > 0$ small, which would generate profits

$$\mu_h G_h(u_{h,n})P_h(u_{h,n}, u_{h,n}) + \mu_l G_l(u_{h,n})P_l(u_{h,n}, u_{h,n}).$$

Using the fact that profits from offer $\mathcal{M}_n$ are at most

$$\frac{\mu_l}{n}(1 - p_l) + \mu_h s_{i,h}(\mathcal{M}_n)P_h(u_{l,n}, u_{h,n}),$$

the firm is making positive profits on high-risk agents,

$$0 < \pi_{i,h}(\mathcal{M}_n) \leq P_h(u_{l,n}, u_{h,n}) \leq P_h(u_{h,n}, u_{h,n}),$$

and the acceptance probability satisfies

$$s_{i,h}(\mathcal{M}_n) \leq G_h(u_{h,n}),$$

we conclude that

$$\|P_h(u_{h,n}, u_{h,n}) - P_h(u_{h,n}, u_{l,n})\| \rightarrow_{n \rightarrow \infty} 0.$$

Since $\lim_n u_{h,n} = \tilde{u}_h$, this implies that $\lim_n u_{l,n} = \underline{u}_l \geq \tilde{u}_h$. High-risk agents must be receiving their utility level with almost no risk; otherwise a firm can profit from a deviation that offers more insurance.

Now we show that $P_l(\underline{u}_l, \tilde{u}_h) = 0$.

First, $P_l(\underline{u}_l, \tilde{u}_h) \geq 0$. If $P_l(\underline{u}_l, \tilde{u}_h) < 0$, then by continuity $P_l(\underline{u}_{l,n}, u_{h,n}) < 0$ for n large. But then one firm can profit by offering $\chi(u_{h,n} + \varepsilon, u_{h,n} + \varepsilon)$ for $\varepsilon > 0$ small. This offer makes weakly more profits from the high-risk agents and guarantees nonnegative profits from low-risk agents.

Second, notice that $P_l(\underline{u}_l, \tilde{u}_h) \leq 0$. Suppose that $P_l(\underline{u}_l, \tilde{u}_h) > 0$.

Also, since

$$\mu_h G_h(u_{h,n})(1 - p_h) \geq \pi_{i,h}(\mathcal{M}_n) \geq \pi_0 - \frac{\mu_l}{n}(1 - p_l),$$

we know that $G_h(u_{h,n}) \geq \frac{\pi_0 - \frac{\mu_l}{n}(1 - p_l)}{(1 - p_h)\mu_h} \rightarrow \frac{\pi_0}{(1 - p_h)\mu_h}$ as $n \rightarrow \infty$. This implies that $G_h(\tilde{u}_h) > 0$.

Case 1.A. Suppose G_h has a mass point at $\tilde{u}_h$. Then there exists offer $\mathcal{M}$ and a firm i , delivering $\mathbf{u} = (u_l, \tilde{u}_h)$ with $u_l \geq \underline{u}_l \geq \tilde{u}_h$ and whose acceptance probability satisfies (ties have to be broken in a way that some firm gets the consumer with probability smaller than 1)

$$s_{i,h}(\mathcal{M}) < G_h(\tilde{u}_h).$$

Notice that $P_h(\mathbf{u}) > 0$ since $u_l \geq \tilde{u}_h$ and $\tilde{u}_h < u(1 - p_h)$.

Also notice that for $\varepsilon > 0$ small,

$$\{(\tilde{u}_h + \varepsilon, \tilde{u}_h + \varepsilon), (u_l + \varepsilon, \tilde{u}_h + \varepsilon)\} \subseteq \text{int}(Y).$$

Hence, following [Lemma 10](#), a firm can find offers that generate profits arbitrarily close to

$$\mu_h G_h(\tilde{u}_h) P_h(\tilde{u}_h, \tilde{u}_h) + \mu_l G_l(\tilde{u}_h) P_l(\tilde{u}_h, \tilde{u}_h)$$

and

$$\mu_h G_h(\tilde{u}_h) P_h(\mathbf{u}) + \mu_l G_l(u_l) P_l(\mathbf{u}).$$

The first profit level leads to a strict improvement if $P_l(\mathbf{u}) < 0$, while the second one leads to a strict improvement if $P_l(\mathbf{u}) \geq 0$. Hence, $\mathbf{u}$ is not optimal for firm i , a contradiction.

Case 1.B. Distribution G_h is continuous at $\tilde{u}_h$. This implies $\tilde{u}_h > \underline{u}_h$, since $G_h(\tilde{u}_h) > 0$. There exists an equilibrium offer $\overline{\mathcal{M}}$ generating utility $\mathbf{u} = (u_h, u_l)$ with $u_l \geq \underline{u}_l$ and $u_h < \tilde{u}_h$, which implies

$$(\underline{u}_l + \varepsilon, \tilde{u}_h) \in \text{int}[\text{co}\{\mathbf{u}, (\underline{u}_l, \underline{u}_l), (u_h, u_h)\}] \subseteq \text{int}(Y).$$

And this implies that $P_l(\underline{u}_l, \tilde{u}_h) = 0$. By way of contradiction, suppose that $P_l(\underline{u}_l, \tilde{u}_h) > 0$. Then, from [Lemma 10](#), we can find $\varepsilon > 0$ sufficiently small such that the following profit can be achieved (using the fact that $\underline{u}_l \geq \tilde{u}_h$):

$$\begin{aligned} & \mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l + \varepsilon, \tilde{u}_h) + \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\underline{u}_l + \varepsilon, \tilde{u}_h) \\ &= \mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l, \tilde{u}_h) + \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\underline{u}_l + \varepsilon, \tilde{u}_h). \end{aligned}$$

By continuity of P_l for ε small enough, we have that

$$\mu_l G_l(\underline{u}_l + \varepsilon) P_l(\tilde{u}_h, \underline{u}_l + \varepsilon) > 0.$$

But $\mu_h G_h(\tilde{u}_h) P_h(\underline{u}_l, \tilde{u}_h)$ weakly higher than the limit of $\pi_i(\mathcal{M}_n)$ as $n \rightarrow \infty$. This means that offer $\mathcal{M}_n$ is strictly dominated, a contradiction. Hence $P_l(\underline{u}_l, \tilde{u}_h) = 0$.

Given that $P_l(\underline{u}_l, \tilde{u}_h) = 0$, then there exists a firm and equilibrium offer $\mathcal{M}$ such that (i) offer $\mathcal{M}$ generates utility $\mathbf{u} = (u_h, u_l)$ with $u_l \geq \underline{u}_l$ and $u_h < \bar{u}_h$, and (ii) $s_{i,l}(\mathcal{M}) > 0$.

But notice that $\pi_{i,l}(\mathcal{M}) \leq P_l(\mathbf{u}) < 0$ since P_l is strictly increasing in u_h whenever $u_h < u_l$. This means that the offer $\mathcal{M}$ cannot be optimal as it is dominated by offer $\chi(u_h + \varepsilon, u_h + \varepsilon)$ for $\varepsilon > 0$ small. This offer yields the same (positive) profits from high-risk agents as $\mathcal{M}$ and guarantees nonnegative profits from low-risk agents.

Case 2. Suppose that $G_l(\underline{u}_l) > 0$.

Case 2.A. Suppose that there exists an equilibrium offer $\mathcal{M}$ generating utility $\mathbf{u} = (u_l, u_h) = (\underline{u}_l, \tilde{u}_h)$ and $G_h(\tilde{u}_h) > 0$.

Suppose that $P_l(\mathbf{u}), P_h(\mathbf{u}) \geq 0$, with this inequality holding strictly for type t . Then consider a firm i such that, when offering $\mathcal{M}$, it has an offer accepted by a type t agent with probability strictly below $G_t(\mathbf{u}_t)$. Then $\mathbf{u} + (\varepsilon, \varepsilon) \in \text{int}(Y)$ and, using Lemma 10, this firm can obtain profit

$$\begin{aligned} & \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, \varepsilon)) + \mu_h G_h(\tilde{u}_h + \varepsilon) P_h(\mathbf{u} + (\varepsilon, \varepsilon)) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(\underline{u}_l) P_l(\mathbf{u}) + \mu_h G_h(\tilde{u}_h) P_h(\mathbf{u}). \end{aligned}$$

And the second term is strictly higher than the profits at $\mathcal{M}$ since the firm has weakly lower profits for each type and attracts both types with weakly higher probabilities (strictly for type t).

Suppose that $P_h(\mathbf{u}), P_l(\mathbf{u}) \leq 0$. This is impossible because offer $\mathcal{M}$ would generate nonpositive profits.

Suppose that $P_h(\mathbf{u}) > 0$ and $P_l(\mathbf{u}) > 0$. Then consider a firm that has offer $\mathcal{M}$ accepted by the high-risk agent with probability strictly lower than $G_h(\tilde{u}_h)^{N-1}$. This firm can profitably deviate by offering $\chi(\tilde{u}_h + \varepsilon, \tilde{u}_h + \varepsilon)$ (it strictly increases profits from high-risk agents and guarantees nonnegative profits from low-risk agents).

Suppose that $P_l(\mathbf{u}) > 0$ and $P_h(\mathbf{u}) < 0$. Consider a firm i that, when offering $\mathcal{M}$, is accepted by a low-risk agent with probability smaller than $G_l(\underline{u}_l)$. From Lemma 9, for $\varepsilon > 0$ small enough, we have that $\mathbf{u} + (\varepsilon, -\varepsilon) \in \text{int}(Y)$. This means that the firm can guarantee profits

$$\begin{aligned} & \mu_l G_l(\underline{u}_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, -\varepsilon)) + \mu_h G_h(\tilde{u}_h - \varepsilon) P_h(\mathbf{u} + (\varepsilon, -\varepsilon)) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(\underline{u}_l) P_l(\mathbf{u}) + \mu_h \left[\lim_{u \nearrow \tilde{u}_h} G_h(u) \right] P_h(\mathbf{u}), \end{aligned}$$

which is strictly higher than profits at $\mathcal{M}$ since it makes less profits, conditional on acceptance by both types, and it attracts the low-risk agent with strictly higher probability while attracting the high-risk agent with weakly lower probability.

Case 2.B. Suppose that there are at least two utility levels $u_h^1 < u_h^2$ such that there are infinitely many offers $\mathcal{M}(\iota)$ that generate utility $\mathbf{u} = (\underline{u}_l, \iota)$ with $\iota \in (u_h^1, u_h^2)$ and ι is not a mass point of G_h .

Consider an arbitrary such utility level $\iota \in (u_h^1, u_h^2)$. The utility profile $(\underline{u}_l, \iota) \in \text{int}(Y)$ since

$$(\underline{u}_l, \iota) \in \text{int}[\text{co}\{(\underline{u}_l, u_h^1), (\underline{u}_l, u_h^2), (u(0), u(0)), (\underline{u}_l + \varepsilon, \underline{u}_l + \varepsilon)\}],$$

which is contained in $\text{int}(Y)$ for $\varepsilon > 0$ sufficiently small.

This means that offer $\mathcal{M}(\iota)$ has to make zero profits on the low-risk agents, i.e., $P_l(\underline{u}_l, \iota) = 0$. Suppose that $P_l(\underline{u}_l, \iota) > 0$ and $P_h(\underline{u}_l, \iota) \geq 0$. Consider a firm that, when offering $\mathcal{M}(\iota)$, has an offer accepted by the low-risk agent with probability strictly below $G_l(\underline{u}_l)$. Then $(\underline{u}_l, \iota) + (\varepsilon, \varepsilon) \in \text{int}(Y)$ for $\varepsilon > 0$ small enough and, from [Lemma 10](#), this firm can obtain profit

$$\begin{aligned} & \mu_l G_l(\underline{u}_l + \varepsilon) P_l((\underline{u}_l, \iota) + (\varepsilon, \varepsilon)) + \mu_h G_h(\iota + \varepsilon) P_h((\underline{u}_l, \iota) + (\varepsilon, \varepsilon)) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(\underline{u}_l) P_l(\underline{u}_l, \iota) + \mu_h G_h(\iota) P_h(\underline{u}_l, \iota), \end{aligned}$$

which is strictly higher than obtained with offer $\mathcal{M}(\iota)$. This follows from the fact that both types are attracted with higher probability strictly for the low-risk consumer, and the firm makes weakly higher profits conditional on acceptance by each agent. The cases (i) $P_l(\underline{u}_l, \iota) > 0$ and $P_h(\underline{u}_l, \iota) < 0$, and (ii) $P_l(\underline{u}_l, \iota) < 0$ and $P_h(\underline{u}_l, \iota) \geq 0$ lead to profitable deviations that exploit the interiority of utility profile $(\underline{u}_l, \iota)$. The case where $P_l(\underline{u}_l, \iota) < 0$ and $P_h(\underline{u}_l, \iota) < 0$ leads to a contradiction since any firm offering such a menu would have nonpositive profits. $\square$

LEMMA 12. *In a symmetric equilibrium, if a firm i makes an offer $\mathcal{M}$ that generates utility profile $\mathbf{u}$, then $\chi_t(\mathbf{u}) \in \mathcal{M}$ and this is the only possible offer accepted by type t from firm i : for any $c \neq \chi_t(\mathbf{u})$,*

$$\mathbb{P}^*[(\tilde{t}, s(\tilde{t}, (\mathcal{M}, \tilde{\mathcal{M}}_{-i}))) = (t, c, i)] = 0.$$

PROOF. First remember that, for each t , problem $P_t(\mathbf{u})$ has a unique solution. This implies that

$$U(c \mid t) = u_t,$$

$$U(c \mid t') \leq u_{t'}$$

implies

$$\Pi(c \mid t) \leq P_t(\mathbf{u})$$

with this inequality holding strictly if $c \neq \chi_t(\mathbf{u})$. This implies that for any offer $\mathcal{M}_i$ delivering utility profile $\mathbf{u}$ and any $(\mathcal{M}_j)_{j \neq i}$,

$$c \in \text{supp}(s_i(t, (\mathcal{M}_k)_k)) \Rightarrow \Pi(c \mid t) \leq P_t(\mathbf{u}),$$

with strict inequality if $c \neq \chi(\mathbf{u})$. Hence, we have that

$$\pi_{i,t}(\mathcal{M}_i) \leq P_t(\mathbf{u}),$$

with this expression holding as a strict inequality if

$$\mathbb{P}^* \{ (\tilde{t}, s(\tilde{t}, (\mathcal{M}, \tilde{\mathcal{M}}_{-i}))) \neq (t, \chi_t(\mathbf{u}, i)) \} > 0. \quad (5)$$

Once again, we proceed by dividing the statement into cases.

Case 1. Assume that $\mathbf{u} \in Y$ satisfies $P_l(\mathbf{u}), P_h(\mathbf{u}) \geq 0$ and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. For $\varepsilon > 0$ sufficiently small, $\mathbf{u} + (\varepsilon, \varepsilon) \in \text{int}(Y)$. Then Lemma 10 implies that any firm can obtain profits

$$\begin{aligned} & \mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, \varepsilon)) + \mu_h G_h(u_h + \varepsilon) P_h(\mathbf{u} + (\varepsilon, \varepsilon)) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h(u_h) P_h(\mathbf{u}). \end{aligned}$$

If (5) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned} & \mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} < \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \\ & \leq \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h(u_h) P_h(\mathbf{u}). \end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, this firm has a profitable deviation.

Case 2. Assume that $\mathbf{u} \in Y$ satisfies $P_l(\mathbf{u}) \geq 0$ and $P_h(\mathbf{u}) \leq 0$, and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. Lemma 9 implies that, for $\varepsilon > 0$ sufficiently small, $\mathbf{u} + (\varepsilon, -\varepsilon) \in \text{int}(Y)$. Then Lemma 10 implies that any firm can obtain profits

$$\begin{aligned} & \mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, -\varepsilon)) + \mu_h G_h(u_h - \varepsilon) P_h(\mathbf{u} + (\varepsilon, -\varepsilon)) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}). \end{aligned}$$

If (5) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned} & \mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} < \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \\ & \leq \mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}). \end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, this firm has a profitable deviation.

Case 3. Assume that $\mathbf{u} \in Y$ satisfies $P_l(\mathbf{u}) \leq 0$ and $P_h(\mathbf{u}) \leq 0$, and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. If (5) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} < \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \leq 0,$$

since $P_t(\mathbf{u}) \leq 0$ for $t = l, h$. Hence firm i would make strictly negative profits by offering $\mathcal{M}$, a contradiction.

Case 4. Assume that $\mathbf{u} \in Y$ satisfies $P_l(\mathbf{u}) \leq 0$ and $P_h(\mathbf{u}) \geq 0$, and consider an equilibrium offer $\mathcal{M}$ that generates this utility profile, i.e., $u_t^{\mathcal{M}} = u_t$ for $t = l, h$. Utility profile

$(u_h + \varepsilon, u_h + \varepsilon) \in \text{int}(Y)$ for $\varepsilon > 0$ sufficiently small. Then Lemma 10 implies that any firm can obtain profits

$$\begin{aligned} & \mu_l G_l(u_h + \varepsilon) P_l(u_h + \varepsilon, u_h + \varepsilon) + \mu_h G_h(u_h + \varepsilon) P_h(u_h + \varepsilon, u_h + \varepsilon) \\ & \rightarrow_{\varepsilon \rightarrow 0} \mu_l G_l(u_h) P_l(u_h, u_h) + \mu_h G_h(u_h) P_h(u_h, u_h). \end{aligned}$$

Now notice that

$$\begin{aligned} P_h(u_h + \varepsilon, u_h + \varepsilon) & \geq P_h(\mathbf{u}) \geq 0, \\ P_l(u_h, u_h) & \geq P_h(u_h, u_h) \geq 0. \end{aligned}$$

If (5) holds, then $s_{i,t}(\mathcal{M}) > 0$ and $\pi_{i,t}(\mathcal{M}_i) < P_t(\mathbf{u})$. Then expected profits with offer $\mathcal{M}$ are given by (we are not indexing everything by $\mathcal{M}$ for brevity)

$$\begin{aligned} & \mu_l s_{i,l} \pi_{i,l} + \mu_h s_{i,h} \pi_{i,h} < \mu_l s_{i,l} P_l(\mathbf{u}) + \mu_h s_{i,h} P_h(\mathbf{u}) \\ & \leq \mu_h s_{i,h} P_h(\mathbf{u}) \\ & \leq \mu_h G_h^-(u_h)^{N-1} P_h(u_h, u_h) \\ & \leq \mu_l G_l(u_l)^{N-1} P_l(u_h, u_h) + \mu_h G_h^-(u_h)^{N-1} P_h(u_h, u_h). \end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, this firm has a profitable deviation. $\square$

LEMMA 13. *In any symmetric equilibrium, firms make nonnegative profits from low-risk agents and nonpositive profits from high-risk agents: any equilibrium utility offer $\mathbf{u}$ satisfies $\mathbf{u} \in \text{int}(Y)$, $u_l \geq u_h$, and*

$$\begin{aligned} \Pi(\chi_l(\mathbf{u})) & \geq 0, \\ \Pi(\chi_h(\mathbf{u})) & \leq 0. \end{aligned}$$

PROOF. Any equilibrium offer $\mathcal{M}$ generating utility profile $\mathbf{u} = (u_l, u_h) \equiv (u_l^{\mathcal{M}}, u_h^{\mathcal{M}})$ satisfies

$$u_t \geq u_t^{\text{RS}}.$$

This means that the consumer always receives offers that are more attractive than the Rothschild–Stiglitz offers. Suppose this is not the case, which implies that $G_t^-(u_t^{\text{RS}}) > 0$ for at least one type $t = l, h$.

Since $(u_l^{\text{RS}}, u_h^{\text{RS}}) \in \text{int}(Y)$, if $\underline{u}_t < u_t^{\text{RS}}$ for a type $t = l, h$, then firms can make profits by offering utility $(u_l^{\text{RS}}, u_h^{\text{RS}}) - (\varepsilon, \varepsilon)$. This offer guarantees positive profits on both types and attracts one of them with positive probability for $\varepsilon > 0$ sufficiently small.

Since utility u_h^{RS} is the generated by an actuarially fair contract with full insurance for the high-risk agents, it follows that $P_h(\mathbf{u}) \leq 0$ for any utility profile $\mathbf{u}$ generated in equilibrium. Also, if there is an equilibrium offer that generates utility $\mathbf{u}$ such that $P_h(\mathbf{u}) < 0$, then at least one firm making such an offer makes strictly negative profits, a contradiction.

Finally, any equilibrium offer generating utility $\mathbf{u} = (u_l, u_h)$ such that $u_h > u_l$ is strictly dominated by offer $\chi(u_l + \varepsilon, u_l + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. This deviating offer attracts high-risk consumers with strictly lower probability and makes strictly higher profits from them, while at the same time attracting low-risk agents with higher probability and loses an arbitrarily small amount of profits. To prove interiority, consider any equilibrium utility offer $\mathbf{u} \in Y$ satisfying $u_l > u_h$. From Lemma 9, it follows that $\mathbf{u} + (\varepsilon, -\varepsilon) \in \text{int}(Y)$ for $\varepsilon > 0$ sufficiently small, which implies that then $\mathbf{u} \in \text{co}\{(u_l, u_l), (u_h, u_h), \mathbf{u} + (\varepsilon, -\varepsilon)\} \subseteq \text{int}(Y)$. $\square$

Proposition 3 follows directly from Lemmas 11, 12, and 13 presented above.

A.5 Proof of Lemma 6

First consider part (i). If $\underline{u}_t < u_t^{\text{RS}}$ for some $t = l, h$, then firms can make profits by offering utility $(u_l^{\text{RS}}, u_h^{\text{RS}}) - (\varepsilon, \varepsilon)$. This offer guarantees positive profits on both types and attracts one of them with positive probability for $\varepsilon > 0$ sufficiently small.

Now we consider part (ii) and divide the proof into steps for clarity.

Step ii.1: For any $n \in \mathbb{N}$, $\underline{\pi}^n \equiv \inf\{P_l(\mathbf{u}) \mid \mathbf{u} \in \text{supp}(G) \text{ and } u_l \geq \underline{u}_l + \frac{1}{n}\} > 0$. Fix $n \in \mathbb{N}$ and assume, by way of contradiction, that there exists a sequence of equilibrium offers $\{\mathbf{u}^n\}$ such that

$$P_l(\mathbf{u}^n) \rightarrow 0.$$

This implies that expected profits from high-risk agents also converges to zero. From Lemma 10, firms can always break indifferences so as to attract high-risk agents with probability $G_h^-(u_h^n)$; hence, profit from high-risk agents is

$$P_h(\mathbf{u}^n)G_h^-(u_h^n) \rightarrow 0.$$

Suppose that $P_h(\mathbf{u}^n) \rightarrow 0$. In this case, we have that $\mathbf{u}^n \rightarrow (u_l^{\text{RS}}, u_h^{\text{RS}})$ since $\max\{P_l(\mathbf{u}), -P_h(\mathbf{u})\}$ has $(u_l^{\text{RS}}, u_h^{\text{RS}})$ as the only zero within the compact set $\{\mathbf{u} \in Y \mid P_h(\mathbf{u}) \leq 0, P_l(\mathbf{u}) \geq 0\}$. But this is a contradiction with $u_l^n \geq \underline{u}_l + \frac{1}{n} > u_h^{\text{RS}}$.

Now suppose that $\liminf P_h(\mathbf{u}^n) > 0$ and $G_h^-(u_h^n) \rightarrow 0$. This implies that $u_h^n \rightarrow \underline{u}_h$ and $G_h(\underline{u}_h) = 0$. Consider a converging subsequence of $\{u_l^n\}$ and let its limit be $\tilde{u}_l$. We know that $P_l(\tilde{u}_l, \underline{u}_h) = 0$ and $\tilde{u}_l \geq \underline{u}_l + \frac{1}{n}$ (which means that $G_l(\tilde{u}_l) \geq G_l(\underline{u}_l + \frac{1}{n}) > 0$). Then, for $\varepsilon > 0$ small, firms can make offer $(\tilde{u}_l - \varepsilon, \underline{u}_h)$ and make profit

$$\mu_l G_l^-(\tilde{u}_l - \varepsilon) P_l(\tilde{u}_l - \varepsilon, \underline{u}_h) > 0,$$

which is positive for $\varepsilon > 0$ sufficiently small, since $P_l(\cdot)$ is decreasing in u_l .

Step ii.2: For any $n \in \mathbb{N}$, G_l is Lipschitz continuous on $[\underline{u}_l + \frac{1}{n}, \bar{u}_l]$, with constant $L_n \equiv \frac{\bar{M}}{\pi^n(N-1)G_l(\underline{u}_l + \frac{1}{n})}$. Consider any $u_l, u'_l \in [\underline{u}_l + \frac{1}{n}, \bar{u}_l]$ such that $u'_l - \varepsilon \leq u_l \leq u'_l$. Consider an equilibrium utility offer $\mathbf{u} = (u_l, u_h)$. If a firm deviates by offering utility $\mathbf{u}' = (u'_l, u_h)$, its profits must not increase:

$$\mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h(u_h) P_h(\mathbf{u}) \geq \mu_l G_l(u'_l) P_l(\mathbf{u}') + \mu_h G_h(u_h) P_h(\mathbf{u}').$$

The fact that low-risk agents receive higher utility, i.e., $u'_l \geq u_l \geq u_h$, implies that $P_h(\mathbf{u}') = P_h(\mathbf{u})$. The inequality becomes

$$[G_l(u'_l) - G_l(u_l)] \leq G_l(u'_l) \frac{[P_l(\mathbf{u}) - P_l(\mathbf{u}')] }{P_l(\mathbf{u})} \leq \varepsilon \frac{\overline{M}}{\underline{\pi}^n},$$

using Lemmas 9 and 10 as well as the fact that $G_l(u'_l) \leq 1$.

Step ii.3. Now, for any $n \in \mathbb{N}$, the restriction of distribution G_l to $[\underline{u}_l + \frac{1}{n}, \overline{u}_l]$ is absolutely continuous, with density g_l^n . Hence, for any bounded nonnegative measurable function f , we have that

$$\int_{\underline{u}_l + \frac{1}{n}}^{\infty} f dG_l = \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f(z) g_l^n(z) dz.$$

Let g_l be the pointwise limit²⁵ of $\{g_l^n\}$ on $(\underline{u}_l, \infty)$ and consider any bounded nonnegative measurable function f .

Define

$$f^n(u) \equiv \mathbf{1}_{[\underline{u}_l, \underline{u}_l + \frac{1}{n})}(u) f(\underline{u}_l) + \mathbf{1}_{[\underline{u}_l + \frac{1}{n}, \infty)}(u) f(u).$$

It follows from the monotone convergence theorem that

$$\lim_{n \rightarrow \infty} \int f^n dG_l = \int f dG_l.$$

But notice that

$$\begin{aligned} \int f^n dG_l &= G_l\left(\underline{u}_l + \frac{1}{n}\right) f(\underline{u}_l) + \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f dG_l \\ &= G_l\left(\underline{u}_l + \frac{1}{n}\right) f(\underline{u}_l) + \int_{\underline{u}_l + \frac{1}{n}}^{\infty} f(z) g_l^n(z) dz. \end{aligned}$$

Taking limits on both sides, and using the dominated convergence theorem once again on the second term yields

$$\int f dG_l = G_l(\underline{u}_l) f(\underline{u}_l) + \int_{\underline{u}_l}^{\infty} f(z) g_l(z) dz.$$

Step ii.4: Suppose that there exists a utility level u_l in the support of G_l such that, for some $\varepsilon > 0$, $G_l(u_l) = G_l(u_l - \varepsilon) > 0$. Consider an offer that generates utility $\mathbf{u} = (u_l, u_h)$ with $u_l > u_h$. This offer is necessarily dominated. From Lemmas 9 and 10, a firm can obtain, by making utility offers close to $\mathbf{u} - (0, -\delta)$, profits arbitrarily close to

$$\mu_l G_l(u_l - \delta) P_l(u_l - \delta, u_h) + \mu_h G_h^-(u_h) P_h(u_h, u_l - \delta),$$

which are strictly higher than equilibrium profits for $\delta < \min\{\frac{\varepsilon}{2}, \frac{u_l - u_h}{2}\}$, since they attract the low-risk agent with the same probability as offer $\mathbf{u}$ and make strictly more profits

²⁵Since $g_l^{n+1}(z) = g_l^n(z)$ for any $z \geq \underline{u}_l + \frac{1}{n}$, this pointwise limit is given by the function $g_l(u) = g_l^{n(u)}(u)$, where $n(u) \equiv \inf\{n' \mid \underline{u}_l + \frac{1}{n'} < u\}$.

from them. Alternatively, any equilibrium utility offer (u_l, u_l) would be dominated by offer $(u_l - \frac{\varepsilon}{2}, u_l - \frac{\varepsilon}{2})$.

Part (iii). Suppose that G_h has a mass point on some utility level $u_h > u_h^{\text{RS}}$. There exists a firm i and equilibrium offer $\mathcal{M}$ that deliver utility $\mathbf{u} = (u_l, u_h)$ that is attracted by the high-risk agent with probability strictly higher than $G_h^-(u_h)$. From Lemmas 9 and 10, for $\varepsilon > 0$ sufficiently small, any firm can obtain profits

$$\mu_l G_l(u_l + \varepsilon) P_l(\mathbf{u} + (\varepsilon, -\varepsilon)) + \mu_h G_h(u_h - \varepsilon) P_h(\mathbf{u} + (\varepsilon, -\varepsilon)),$$

which converge, as $\varepsilon \rightarrow 0$, to

$$\mu_l G_l(u_l) P_l(\mathbf{u}) + \mu_h G_h^-(u_h) P_h(\mathbf{u}).$$

This profit is strictly higher than the equilibrium profit since it attracts high-risk agents with strictly lower probability and $P_h(\mathbf{u}) < 0$ since $u_l > u_l^{\text{RS}}$.

A.6 Proof of Proposition 5

Since I consider variation in μ_l explicitly, I acknowledge dependence of each variable on μ_l as a superscript in the notation, as in $\bar{k}^{\mu_l}$.

First notice that, for each $k \in R$, $\gamma^{\mu_l}(k)$ is the solution $\gamma = (\gamma_1, \gamma_0)$ (with $\gamma_1 \geq \gamma_0$) to the system

$$A_1(\gamma, \mu_l, k) \equiv \begin{bmatrix} (1 - \mu_l)(-k) + \mu_l \Pi(\gamma | l) \\ U(\gamma | h) - u(1 - p_h + k) \end{bmatrix} = 0.$$

Then, since $\frac{\partial A_1}{\partial \gamma}$ has full rank and $A_1(\cdot)$ is continuously differentiable, it follows that $\gamma^{\mu_l}(k)$ is continuously differentiable in (μ_l, k) . It is simple to show that $\frac{\partial \gamma_1^{\mu_l}(k)}{\partial \mu_l} > 0$ and $\frac{\partial \gamma_0^{\mu_l}(k)}{\partial \mu_l} < 0$. From Lemma 2, I know that, if $\bar{k}^{\mu_l} > 0$, $\bar{k}^{\mu_l}$ is defined implicitly as the solution to the equation

$$\begin{aligned} A_2(k, \mu_l) &\equiv u'(1 - p_h + k)(1 - p_l) \left[\frac{1}{u'(\gamma_1^{\mu_l}(k))} - \frac{1}{u'(\gamma_0^{\mu_l}(k))} \right] \\ &\quad - \frac{\mu_h}{\mu_l} \frac{1}{p_h} \left[\frac{1 - p_l}{p_l} - \frac{1 - p_h}{p_h} \right] \\ &= 0. \end{aligned}$$

But we already showed that $\gamma^{\mu_l}(k)$ is continuously differentiable in (μ_l, k) and, therefore, so is A_2 . Since $\frac{\partial A_2(k, \mu_l)}{\partial k} < 0$, $\bar{k}^{\mu_l}$ is also continuously differentiable. Moreover, it follows from $\frac{\partial A_2(k, \mu_l)}{\partial \mu_l} > 0$ that $\frac{\partial \bar{k}^{\mu_l}}{\partial \mu_l} > 0$. Finally, for any $\mu_l(0, 1)$ and $k \in [0, p_h - \bar{p}^{\mu_l}]$, $A_2(k, \mu_l) > 0$ for μ_l sufficiently high. Hence, since $\bar{p}^{\mu_l} \rightarrow_{\mu_l \rightarrow 1} p_l$, $\bar{k}^{\mu_l} \rightarrow p_h - p_l$ as μ_l converges to 1.

Also, notice that, after substitution,

$$\begin{aligned} \phi^{\mu_l}(k) \equiv & \left(\frac{p_h p_l}{p_h - p_l} \left\{ u'(1 - p_h + k)(1 - p_l) \left[\frac{1}{u'(\gamma_1^{\mu_l}(k))} - \frac{1}{u'(\gamma_0^{\mu_l}(k))} \right] \right. \right. \\ & \left. \left. - \frac{\mu_h}{\mu_l} \frac{1}{p_h} \left[\frac{1 - p_l}{p_l} - \frac{1 - p_h}{p_h} \right] \right\} \right) / (\Pi(\gamma^{\mu_l}(k) | l)). \end{aligned} \quad (6)$$

For any $(\mu_l, k) \in (0, 1) \times [0, p_h - \bar{p}]$, $\Pi(\gamma^{\mu_l}(k) | l)$ satisfies

$$(1 - \mu_l)(-k) + \mu_l \Pi(\gamma^{\mu_l}(k) | l) = 0.$$

This implies that $\frac{\partial \Pi(\gamma^{\mu_l}(k) | l)}{\partial \mu_l} < 0$ and that, for any $k \in [0, \bar{k}^{\mu_l}]$,

$$\frac{\partial \phi^{\mu_l}}{\partial \mu_l} > 0.$$

Also, since the numerator in (6) is increasing in μ_l and the denominator converges to zero as $\mu_l \rightarrow 1$, we know that for all $k \in [0, p_h - p_l]$,

$$\phi^{\mu_l}(k) \rightarrow \infty.$$

Now consider $\mu'_l > \mu_l$. In the case $\bar{k}^{\mu_l} = 0$, the result is trivial; otherwise we have from the differential equation (3) satisfied by $[F^{(\mu_l)}]^{N-1}$ that for any $k \in [0, \bar{k}^{\mu_l}]$,

$$\begin{aligned} \left[\frac{F^{(\mu_l)}(k)}{F^{(\mu'_l)}(k)} \right]^{N-1} &= \exp \left[\int_k^{\bar{k}^{\mu'_l}} \phi^{\mu'_l}(z) dz - \int_k^{\bar{k}^{\mu_l}} \phi^{\mu_l}(z) dz \right] \\ &= \exp \left[\int_{\bar{k}^{\mu_l}}^{\bar{k}^{\mu'_l}} \phi^{\mu'_l}(z) dz + \int_k^{\bar{k}^{\mu_l}} [\phi^{\mu'_l}(z) - \phi^{\mu_l}(z)] dz \right] > 1. \end{aligned}$$

Monotone convergence, together with monotonicity of $\bar{k}^{\mu_l}$ and $\phi^{\mu_l}(\cdot)$, implies that for any $k \in [0, p_h - p_l]$,

$$\mu_l \mapsto \int_k^{\bar{k}^{\mu_l}} \phi^{\mu_l}(z) dz \quad \text{is continuous}$$

and

$$\int_k^{\bar{k}^{\mu_l}} \phi^{\mu_l}(z) dz \rightarrow \infty, \quad \text{as } \mu_l \rightarrow 1.$$

Hence, it follows that, for any $k \in [0, p_h - p_l]$, $F^{\mu_l}(k)$ is continuous in μ_l and $F^{\mu_l}(k) \rightarrow 0$ as $\mu_l \rightarrow 1$.

The results for $\bar{F}^{\mu_l}$ follow from $\bar{F}^{\mu_l} = (F^{\mu_l})^N$.

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