Auction design in the presence of collusion

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We study a problem of optimal auction design in the realistic case in which the players can collude both on the way they play in the auction and on their participation decisions. Despite the fact that the principal's opportunities for extracting payments from the agents in such a situation are limited, we show how the asymmetry of information between the colluding agents can be used to reduce the revenue losses from collusion. In a class of environments we show that the principal is even able to achieve the same revenue as when the agents do not collude. For cases in which it is not possible to do so we provide an optimal mechanism in the class of mechanisms with linear and symmetric menus and discuss the potential benefits of using asymmetric and nonlinear mechanisms. To address the problem of multiplicity of equilibria we show how the optimal mechanisms can be implemented as uniquely collusion-proof mechanisms.

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JEL classification. C61, D44, D82, L41.

1. Introduction

In many environments a revenue-maximizing method of selling scarce goods is to organize an auction.[1] However, since an auction exploits competition between the agents to raise the revenue of the seller, it creates strong incentives for collusion between the agents against the seller. In this paper we study a problem of optimal auction design in the realistic case in which the players not only can collude on the way they play in the auction, but also can coordinate their decisions regarding whether to participate in the auction.

Previous studies of collusion in auctions focus on the optimal organization of a cartel in standard auctions. For example, Graham and Marshall (1987) study collusion in

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[1] See for example Myerson (1981).

second-price auctions and McAfee and McMillan (1992) study collusion in first-price auctions. They show that in standard auctions the agents are able to collude efficiently provided that they can exchange side payments. In such auctions with collusion a seller never gets paid more than the reservation price. These studies also investigate some simple responses of the seller to collusion, the most common of which is raising the reservation price. This strategy, although helpful in reducing revenue loss, does not render collusion completely ineffective. Other suggested responses for the seller consist of various ways of interfering with the cartel's enforcement mechanism, which are not always available to the seller.

The next logical step in exploring the question of the seller's optimal response to collusion is to go beyond the standard auction formats and to formulate the seller's problem as one of mechanism design. Laffont and Martimort (1997, 2000) pursue this approach to study the principal's response to collusion in problems of regulating firms and providing a public good. A valuable insight of their work is to emphasize that the agents may often fail to collude efficiently. In order to ensure the agents' participation in the cartel, the collusive agreement must provide the agents with higher payoffs than they could obtain by refusing to participate in the cartel and playing in the principal's mechanism noncooperatively. These constraints, given that the agents have private information, may make it impossible for the cartel to operate efficiently. A principal's optimal mechanism should thus take advantage of the presence of these constraints and the resulting inefficiency of the collusive agreement to mitigate the effects of collusion on revenue.

One intriguing result established by Laffont and Martimort (1997) is their proof of the existence of an optimal mechanism that manages to eliminate all the effects of collusion. Che and Kim (2006) significantly generalize this result and provide a compelling intuition for it. If the agents' decisions on whether to participate in the principal's mechanism precede the formation of the cartel, then the principal can "sell the allocation problem to the cartel" at a constant price and attain the same expected payoff as in the absence of collusion. The cartel becomes the residual claimant of the total surplus, and the constraints the cartel is facing force it to implement the allocation desired by the principal. Though the mechanism of Che and Kim (2006) is robust in a number of ways, it relies on the timing of the agents' decisions to participate in the principal's mechanism. Their mechanism fails to work if the agents have a chance to share their private information before they decide whether to participate, because then they can "buy the allocation problem" from the principal only in cases when it is worthwhile for the cartel. In Section 2 we use an example to discuss the mechanism proposed by Che and Kim (2006) and its reliance on the timing of participation decisions.

We study the problem of optimal auction design when the players can collude both on the way they play in the auction and on their participation decisions. Notice that the principal's opportunities for extracting payments from the agents in this situation are seriously limited. First, it is no longer possible for the principal to collect payments from the agents but not allocate the good to any of them. In such a case the agents can collectively decide against participating in the principal's mechanism and thus avoid any payments. Second, the cartel is able to coordinate the behavior of the agents in the principal's mechanism and thus to minimize the total expected payments for any given allocation. Hence, it is impossible for the total expected payment the principal receives for a given allocation in one case to differ from the total expected payment for the same allocation in another case, because the cartel would always choose the lowest total expected payment for a given allocation. This is in contrast to any standard auction without collusion where the price at which a good is sold depends on the submitted bids. So in a sense the cartel behaves as a single agent. This reasoning is formalized in Section 4.1, where for every given mechanism of the principal we derive a corresponding menu of feasible allocations and the minimal costs for the cartel of achieving these allocations.

However, a collusive mechanism must also provide the agents with payoffs that are no lower than those they expect to get if they refuse to participate in the collusive mechanism and play the principal's mechanism noncooperatively. This task of providing these payoffs is not always simple given that the agents have private information. So in this sense the cartel does not act as a single agent, but rather performs as a collective decision-making institution that must be ratified by every type of every agent. Hence, by choosing a mechanism the principal determines the outside options of the agents in the cartel problem, which constrain the set of feasible collusive mechanisms. Theorem 1 describes a class of possible resulting allocations in the game between the principal and the cartel. In Theorem 2 we show that for every allocation from this class there exists a principal's mechanism that ensures that this allocation is achieved.

A carefully designed mechanism enables the principal to limit the scope of collusion and reduce revenue losses. In Theorem 3 we show that in a class of environments the principal is in fact able to eliminate all losses from collusion and achieve the same revenue as in the case that the agents do not collude. An example of such a mechanism is provided in Section 2. For cases when it is not possible to do so, Theorem 4 provides an optimal mechanism in the class of mechanisms with linear and symmetric menus; the potential benefits of using asymmetric and nonlinear mechanisms are discussed with help of examples.

Che and Kim (2007) independently study the same problem and obtain many of the results of this paper. Our results on the optimal collusion-proof mechanisms apply to cases in which the principal has a single good for sale, the agents' valuations for the good are independently and identically distributed, and all agents participate in the cartel. Che and Kim (2007) in addition present results on asymmetric agents and on partial cartels. We consider both a case when the principal can prohibit reallocation of the good between the colluding agents and when she cannot, while Che and Kim (2007) consider only the latter case. Another point of difference is that in addition to solving for the optimal weakly collusion-proof mechanisms we also introduce a new notion of uniquely collusion-proof mechanisms to address the important problem of the multiplicity of equilibria, and in Theorem 5 we show how the optimal mechanisms can be implemented as uniquely collusion-proof mechanisms. There are also methodological differences between the two papers, which are discussed in Section 6.4.[2]

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[2]Dequiedt (2007), Quesada (2005), and Mookherjee and Tsumagari (2004) also consider the possibility

The rest of the paper is organized as follows. A motivating example is considered in Section 2. The model is presented in Section 3. The problem of the cartel is studied in Section 4. In Section 5 we characterize the principal's optimal mechanisms in the presence of collusion. We discuss several extensions and the related literature in Section 6, and conclude in Section 7. All proofs are relegated to the Appendix unless stated otherwise.

2. Example

Suppose a principal wants to auction a single good to two agents whose valuations are independently uniformly distributed on $[0,1]$.[3] If the agents do not collude then according to Myerson (1981) a mechanism that maximizes the expected revenue of the seller has the following properties: (i) the good must be allocated to the agent with the highest valuation as long as it exceeds the cutoff of $\frac{1}{2}$, and be withheld otherwise; (ii) the expected payoff of the agent with the valuation 0 must be equal to zero. The expected Myerson revenue is $\Pi^* = \frac{5}{12}$. Below we discuss three implementations of the optimal mechanism, all of which perform equally well when there is no collusion, but which yield drastically different expected revenues when the agents collude.

The first mechanism is a second-price auction with the reservation price $\frac{1}{2}$. This auction has a symmetric equilibrium in (weakly) dominant strategies where each agent i bids his valuation θ_i if it exceeds $\frac{1}{2}$, and stays out otherwise. This equilibrium results in the Myerson allocation, and the seller receives $\max\{\frac{1}{2}, \min\{\theta_1, \theta_2\}\}$ if she makes a sale, and 0 otherwise.

Graham and Marshall (1987) show that the second-price auction is susceptible to collusion. They construct an incentive compatible "preauction knockout" collusive mechanism under which the agents buy the good from the seller at the reservation price $\frac{1}{2}$ and allocate the good to the agent with the highest valuation if and only if the highest valuation exceeds the reservation price $\frac{1}{2}$. This collusive mechanism results in the Myerson allocation, but the principal's expected revenue falls below the Myerson level. She now receives only the reservation price $\frac{1}{2}$ in case of a sale, while the remainder is captured and redistributed between the colluding agents.

In the second mechanism each agent is charged a fixed fee of $\frac{1}{2}\Pi^*$, and then the agents simultaneously decide whether to bid for the good or to stay out. If both agents stay out then neither of them gets the good and there are no additional payments. If agent i bids and the other agent stays out, then agent i gets the good and pays his bid to his opponent. If both agents submit bids, then the highest bidder gets the good and the agents pay their bids to each other. There is a symmetric equilibrium where each agent i bids $\frac{1}{2}(\theta_i)^2 + \frac{1}{8}$ if his valuation θ_i exceeds $\frac{1}{2}$, and stays out otherwise. This equilibrium results in the Myerson allocation, and the principal receives only the sum of the fixed fees, which is equal to Π^* regardless of the agents' actions taken in the bidding game.

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of collusion on participation decisions. We discuss the relation to these papers in Section 6.4.

[3]Throughout the paper we use feminine pronouns for the principal and masculine pronouns for the agents.

Che and Kim (2006) show that this mechanism renders collusion ineffective and achieves the Myerson revenue if collusion takes place after the agents have agreed to participate in the mechanism. Indeed, under such a scenario collusion occurs "too late": by agreeing to participate the agents have already committed to pay the fixed fees that provide the principal with the Myerson revenue regardless of the agents' subsequent actions. But the only way for the cartel to fulfill this commitment, while still inducing the agents' participation in the collusive scheme and without breaking the budget, is to implement the Myerson allocation.

However, this mechanism fails to be collusion-proof if collusion takes place before the agents have agreed to participate in the mechanism. If the agents share the information about their valuations beforehand, then they can refuse to participate in the mechanism if both valuations are smaller than the total fixed fee Π^*. The principal's expected revenue then falls below the Myerson level, because she receives the total fixed fee Π^* only in case of a sale.

In the third mechanism the agents simultaneously decide whether to bid for the good or to stay out. If agent i stays out, then he does not get the good and his payment is zero. If agent i bids and the other agent stays out, then agent i gets the good and pays the price $\frac{5}{9}$ to the principal regardless of his bid. If both agents submit bids, then the highest bidder gets the good and pays his bid to his opponent, while the loser pays $\frac{5}{9}$ to the principal. There is a symmetric equilibrium where each agent i bids $\frac{1}{3}\theta_i + \frac{13}{36}$ if his valuation θ_i exceeds $\frac{1}{2}$, and stays out otherwise. This equilibrium results in the Myerson allocation, and the principal receives $\frac{5}{9}$ if she makes the sale, and 0 otherwise.

In this paper we show that the third mechanism renders collusion ineffective and achieves the Myerson revenue even when collusion takes place before the agents have agreed to participate in the mechanism. The Myerson allocation turns out to be *cartel interim efficient* when the cartel is facing such a mechanism: any alternative feasible allocation necessarily makes some types of some agents worse off and thus is vetoed. For example, consider a collusive mechanism that maximizes the sum of the agents' ex ante expected payoffs: the agents buy the good from the seller at the price $\frac{5}{9}$ and allocate the good to the agent with the highest valuation if and only if the highest valuation exceeds the price $\frac{5}{9}$. One can prove that such a collusive mechanism provides agents who have sufficiently high valuations with expected payoffs lower than those they expect to get through noncooperative play in the principal's mechanism and thus is vetoed.

3. Model

There is one principal who owns a good, and $n \geq 2$ agents. Each agent i has a valuation θ_i for the good, which is known only to him. Valuations are identically and independently distributed according to a continuous cumulative distribution function F with support $[\underline{\theta}, \overline{\theta}]$, where $0 \leq \underline{\theta} < \overline{\theta} < \infty$, and an everywhere positive differentiable density f. This distribution is common knowledge. We require the distribution to satisfy a standard condition on the hazard rates.

Condition 1. The distribution F satisfies

$$\frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right)\leq 0\leq \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \text{ for every } \theta\in(\underline{\theta},\overline{\theta}).$$

All players have quasi-linear utilities. Agent i's utility is $\theta_i p_i - t_i$, where p_i is his probability of getting the good and t_i is his payment. The principal's utility is $\sum_{i=1}^n t_i$.

The principal offers a *grand mechanism* Γ, which consists of a set $M=\times_{i=1}^n M_i$ of message profiles, an allocation rule $\widetilde{p}: M\rightarrow\Sigma=\{\sigma\in\mathbb{R}_+^n \mid \sum_{i=1}^n \sigma_i\leq 1\}$, and a payment rule $\widetilde{t}: M\rightarrow\mathbb{R}^n$. For every i, M_i is the set of messages available to i. We assume that each M_i contains a special message $\varnothing$ that indicates his rejection of the grand mechanism. In case agent i sends the message $\varnothing$, he does not receive the good and there is no payment. An allocation rule $\widetilde{p}$ associates with each message profile a vector of probabilities, $\widetilde{p}_i(m)$ being the probability that agent i is assigned the good when the message profile is m. A payment rule $\widetilde{t}$ associates with each message profile a vector of payments, $\widetilde{t}_i(m)$ being the payment that agent i pays when the message profile is m.[4]

An uninformed third party, a *cartel*, proposes a *side mechanism* S. From the revelation principle there is no loss of generality in assuming that S is a direct mechanism. So the side mechanism consists of a set $\Theta=[\underline{\theta},\overline{\theta}]^n$ of profiles of type reports, a message reporting function $\phi:\Theta\rightarrow\Delta M$ (where ΔM is the space of probability distributions over M), and a side payment function $y:\Theta\rightarrow\mathbb{R}^n$. A message reporting function ϕ associates with each profile of reported types a probability distribution over message profiles reported to the grand mechanism. A side payment function y associates with each profile of reported types a vector of side payments, $y_i(\theta)$ being the payment that agent i receives from the cartel when the type profile is θ. The side payment function is assumed to be ex post budget balanced: $\sum_{i=1}^n y_i(\theta)\leq 0$ for every $\theta\in\Theta$. The side mechanism cannot be observed by the principal, and the principal cannot verify what payments took place between the agents.

Notice that in this formulation reallocations of the good between the agents are not allowed. In Section 6.2 we show that most of the results of the paper remain true even when the principal has no control over reallocations.

The timing of the game is as follows.

Timing of the game

1. The agents learn their valuations.
2. The principal offers a grand mechanism Γ.
3. The cartel proposes a side mechanism S.
4. The agents simultaneously decide whether to accept or reject the side mechanism.

   If all agents accept S, then go to step 5. Otherwise go to step 5′.

[4]The principal never benefits from randomized payments because all the players' payoffs are quasi-linear. Thus there is no loss of generality in restricting attention to deterministic payment rules.

5. Each agent i reports a type $\widehat{\theta}_i$ to the side mechanism.

6. A message profile m is determined according to $\phi(\widehat{\theta})$, where $\widehat{\theta} = (\widehat{\theta}_1, \ldots, \widehat{\theta}_n)$, and is sent to the grand mechanism.

7. Allocations $\widetilde{p}(m)$, payments $\widetilde{t}(m)$, and side payments $y(\widehat{\theta})$ are enforced.

5′. Each agent i reports a message m_i to the grand mechanism.

6′. Allocations $\widetilde{p}(m)$ and payments $\widetilde{t}(m)$, where $m = (m_1, \ldots, m_n)$, are enforced.

Notice that if the agents agree to collude, then they respond to the principal's offer after communicating with each other, because a message reporting function ϕ can prescribe sending a rejection message $\varnothing$ on behalf of each of the agents. This formulation seems natural: agents can coordinate not only on the way they play the grand mechanism, but also on their participation decisions.

Let us define a *collusive allocation function* $p : \Theta \to \Sigma$ and a *collusive payment function* $t : \Theta \to \mathbb{R}^n$ in order to describe what the cartel can do when facing a given grand mechanism Γ. To state the problem of the cartel formally it is convenient to separate the constraints describing feasibility and budget balancedness of the collusive allocation and payment functions for a given grand mechanism from the rest of the constraints the cartel is facing. For any given profile θ of the reported types, a randomization over the message profiles to be sent to the grand mechanism is performed according to $\phi(\theta)$. Each message m results in a vector of allocations $\widetilde{p}(m)$; the expectation over these allocations according to $\phi(\theta)$ gives $p(\theta)$.[5] Similarly, the expectation over vectors of payments $\widetilde{t}(m)$ the agents pay, together with a vector of budget balanced side payments $y(\theta)$ the agents receive, determines $t(\theta)$.

Definition 1. A pair of collusive allocation and payment functions $(p, t) : \Theta \to \Sigma \times \mathbb{R}^n$ is *feasible and budget balanced* in the grand mechanism $\Gamma = \langle M, \widetilde{p}, \widetilde{t} \rangle$ if there exists a side mechanism $S = \langle \Theta, \phi, y \rangle$ such that

(i) $p_i(\theta) = E_{\phi(\theta)}[\widetilde{p}_i(m)]$ and $t_i(\theta) = E_{\phi(\theta)}[\widetilde{t}_i(m)] - y_i(\theta)$ for every i and $\theta \in \Theta$, where $E_{\phi(\theta)}[\cdot]$ denotes the expectation according to $\phi(\theta)$ over M

(ii) $\sum_{i=1}^n y_i(\theta) \leq 0$ for every $\theta \in \Theta$.

The remaining constraints of the cartel's problem, as well as the objective of the cartel, can be stated directly in terms of the collusive allocation and payment functions (p, t). Define the interim expected payoff of agent i of type θ_i from participation in the

[5] In some environments free disposal of the good should be allowed. For example, free disposal seems plausible for antiques auctions, but not plausible for procurement contracts auctions. It is straightforward to incorporate free disposal in the model of collusion, but it makes the notation more cumbersome without adding much value.

side mechanism and following a truthful strategy by

$$U_i(\theta_i) = E_{\theta_{-i}}[\theta_i p_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})].$$

The pair of collusive allocation and payment functions (p, t) must be incentive compatible to ensure truthful type reporting by the agents.[6] It must also be individually rational: the interim expected payoffs from participation in the side mechanism have to be at least as large as the interim expected payoffs from noncooperative play in the grand mechanism following the rejection of the side mechanism. We denote the interim expected payoff of agent i of type θ_i from nonparticipation by $V_i(\theta_i)$, and call $V_1, \dots, V_n$ the *outside options* in the cartel's problem. Note that in an equilibrium where all agents are supposed to accept the side mechanism a rejection by agent i may change the other agents' beliefs about the type of agent i, and, thus, the outside option of agent i may depend on these revised beliefs.

The objective of the cartel is to maximize the weighted interim expected payoffs of the agents. We allow for heterogenous welfare weights over agents and different types of agents, which may reflect an asymmetric distribution of bargaining power in the cartel. The welfare weights are described by non-negative, non-decreasing, right-continuous functions $\omega_1, \dots, \omega_n$, where $\omega_i(\theta_i)$ is the fraction of agent i's welfare weight concentrated on i's types that are weakly lower than θ_i. To make the problem of the cartel non-trivial we assume

$$\sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} d\omega_i(\theta_i) > 0,$$

where by the integral $\int_a^b$ with respect to ω_i we mean the Lebesgue–Stieltjes integral $\int_{[a,b]}$ with respect to ω_i.

The problem of the cartel facing the grand mechanism $\Gamma = \langle M, \widetilde{p}, \widetilde{t} \rangle$ with outside options $V_1, \dots, V_n$ is stated below.

Program C: $$\max_{(p,t)} \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, d\omega_i(\theta_i) \quad \text{subject to}$$

Feasibility and Budget Balance: (p, t) is feasible and budget balanced in the grand mechanism Γ

Incentive Compatibility: $U_i(\theta_i) \geq E_{\theta_{-i}}[\theta_i p_i(\widehat{\theta}_i, \theta_{-i}) - t_i(\widehat{\theta}_i, \theta_{-i})]$ for every i and $\theta_i, \widehat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$

Individual Rationality: $U_i(\theta_i) \geq V_i(\theta_i)$ for every i and $\theta_i \in [\underline{\theta}, \overline{\theta}]$.

Notice that the individual rationality constraints require that every type of every agent be willing to participate in the cartel in order for collusion to take place. We comment on the generality of setting up the cartel's problem in this way in Section 6.3.

[6]According to the revelation principle there is no loss of generality in focusing on the truth-telling equilibrium.

4. The cartel's problem

4.1 *Feasibility and budget balance*

We introduce the concept of a *menu* to describe feasible and budget balanced collusive outcomes for a given grand mechanism $\Gamma = \langle M, \tilde{p}, \tilde{t} \rangle$. A menu describes the set of allocations the cartel can achieve, as well as the minimal cost of achieving any particular allocation in the grand mechanism Γ. Define the set of allocations that the cartel can achieve by sending various message profiles in the grand mechanism:

$$\Sigma_1 = \{p \in \Sigma \mid \text{there exists } m \in M \text{ such that } p = \tilde{p}(m)\}.$$

Define the function $c_1 : \Sigma_1 \to \mathbb{R}$ to be the lowest bound for the aggregate cost for the cartel at which any given allocation p can be purchased:

$$c_1(p) = \inf_{\{m:\, \tilde{p}(m)=p\}} \sum_{i=1}^{n} \tilde{t}_i(m) \quad \text{for every } p \in \Sigma_1.$$

To avoid potential problems with the existence of a solution to the cartel's problem we impose the following assumption.

Condition 2. For every sequence of messages $\{m^k\}$ such that

$$\lim_{k\to\infty} \left(\tilde{p}(m^k), \sum_{i=1}^{n} \tilde{t}_i(m^k) \right)$$

exists, there exists a message $m \in M$ such that $\left(\tilde{p}(m), \sum_{i=1}^{n} \tilde{t}_i(m)\right)$ is equal to the limit.

Since the cartel can randomize over the message profiles it sends to the principal, the set of feasible allocations may be larger than Σ_1. Define Σ^* to be the convex hull of Σ_1:

$$\Sigma^* = \text{co}(\Sigma_1).$$

Define a function $c : \Sigma^* \to \mathbb{R}$ to be the greatest convex function that is pointwise weakly lower than c_1:[7]

$$c(p) = \text{vex}(c_1(p)) \quad \text{for every } p \in \Sigma^*.$$

As an illustration, let the grand mechanism be the first- or second-price auction with a common reservation price $R \geq 0$. For given bids the allocation can be either null, or a transfer of the good to a particular agent, or a fair lottery between two or more agents in case of a tie. This is the set Σ_1. The minimal cost for the cartel (c_1) when the good is transferred from the principal is R, and is 0 otherwise. The cartel can achieve any allocation in Σ by appropriate randomization over submitted bids. Thus the set Σ^* is the whole feasible set Σ. It follows that $c(p) = R \sum_{i=1}^{n} p_i$. We return to this class of menus in Section 5.

We say that a pair (Σ^*, c) for a given grand mechanism Γ is the *menu of the grand mechanism* Γ. The next result characterizes feasible and budget balanced collusive outcomes in a given grand mechanism Γ using the concept of a menu.

[7]See for example Section 5 in Rockafellar (1970).

Lemma 1. *Consider a grand mechanism* Γ *with menu* (Σ^*, c). *A pair of collusive allocation and payment functions* $(p,t):\Theta \to \Sigma \times \mathbb{R}^n$ *is feasible and budget balanced in* Γ *if and only if*

(i) $p(\theta) \in \Sigma^*$ *for every* $\theta \in \Theta$

(ii) $t(\theta)$ *is such that* $\sum_{i=1}^n t_i(\theta) \geq c(p(\theta))$ *for every* $\theta \in \Theta$.

This result provides a considerably more tractable characterization of feasible and budget balanced collusive allocation and payment functions than the one given in Definition 1. The set of achievable allocations and feasible payments in any grand mechanism is succinctly described by the menu of this grand mechanism.

Notice that the cartel can always achieve the null allocation and avoid any payments by sending the rejection message $\varnothing$ to the grand mechanism on behalf of every agent. Hence, in any grand mechanism Γ the menu (Σ^*, c) must satisfy $\mathbf{0} \in \Sigma^*$ and $c(\mathbf{0}) \leq 0$. This important restriction on the menus of grand mechanisms can be viewed as due to the possibility that the agents can collude on their participation decisions.

4.2 *Simplifying the cartel's problem*

In this section we reformulate and simplify the cartel's problem. So far we have stated the problem of the cartel as a problem of the choice of collusive allocation and payment functions $(p,t):\Theta \to \Sigma \times \mathbb{R}^n$. It turns out that now it is more convenient to think of the cartel as choosing an allocation function $p:\Theta \to \Sigma$ and interim payoffs $U_1,\dots,U_n$ for the agents, where $U_i:[\underline{\theta},\overline{\theta}] \to \mathbb{R}$.

Proposition 1. *The problem of the cartel facing a grand mechanism* Γ *with menu* (Σ^*, c) *and outside options* $V_1,\dots,V_n$ *(Program C), is equivalent to the following problem.*

Program C:*
$$\max_{(p,U_1,\dots,U_n)} \sum_{i=1}^n \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, d\omega_i(\theta_i) \quad \textit{subject to}$$

F: $p(\theta) \in \Sigma^*$ *for every* $\theta \in \Theta$

IC: $E_{\theta_{-i}}[p_i(\theta_i,\theta_{-i})]$ *is non-decreasing in* θ_i *for every* i, *and*

$$U_i(\theta_i) = U_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i,\theta_{-i})]\, d\widetilde{\theta}_i \quad \textit{for every } i \textit{ and } \theta_i \in [\underline{\theta},\overline{\theta}]$$

BB: $$\sum_{i=1}^n U_i(\underline{\theta}) \leq E\left[\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right]$$

IR: $U_i(\theta_i) \geq V_i(\theta_i)$ *for every* i *and* $\theta_i \in [\underline{\theta},\overline{\theta}]$.

The first set of constraints, denoted by F, is just the set of feasibility constraints provided in part (i) of Lemma 1. The last set of constraints, denoted by IR, is the set of individual rationality constraints that ensure participation in the collusive side mechanism. In the three lemmas below we show that the set of incentive compatibility constraints can be equivalently represented by condition IC, and the set of budget balance constraints, provided in part (ii) of Lemma 1, can be equivalently represented by the single constraint BB.

Notice that the objective of the cartel, as well as the incentive compatibility and individual rationality constraints, depend only on the interim expected payment $E_{\theta_{-i}}[t_i(\theta_i,\theta_{-i})]$ for every i and θ_i, rather than on the ex post payment $t_i(\theta)$ for every i and θ. Hence, from the point of view of the cartel, any two pairs of collusive allocation and payment functions are equivalent if their allocation functions and interim expected payments coincide. This observation allows us to replace the ex post budget balance constraints given in Lemma 1 with a single ex ante budget balance constraint. We say that a pair of collusive allocation and payment functions $(p,t):\Theta\to\Sigma\times\mathbb{R}^n$ in a grand mechanism Γ with menu (Σ^*,c) is *ex ante budget balanced* if

$$E_\theta\left[\sum_{i=1}^n t_i(\theta)-c(p(\theta))\right]\geq 0.$$

Lemma 2. *Consider an ex ante budget balanced pair of collusive allocation and payment functions* $(p,t):\Theta\to\Sigma\times\mathbb{R}^n$ *in a grand mechanism* Γ *with menu* (Σ^*,c). *Then there exists a budget balanced pair of collusive allocation and payment functions* $(p,t'):\Theta\to\Sigma\times\mathbb{R}^n$ *such that*

$$E_{\theta_{-i}}[t_i'(\theta_i,\theta_{-i})]=E_{\theta_{-i}}[t_i(\theta_i,\theta_{-i})] \quad \textit{for every } i \textit{ and } \theta_i\in[\underline{\theta},\overline{\theta}].$$

Next we simplify the set of incentive compatibility constraints. The argument is standard (see for example Myerson 1981), and we state the result without proof.

Lemma 3. *The cartel's incentive compatibility constraints are equivalent to condition IC in Program C* given in Proposition 1.*

According to Lemma 3, for every agent i the allocation function p_i determines the individual payoff U_i and the expected payment up to a constant $U_i(\underline{\theta})$, which is the payoff of the lowest type. Next we show that the sum of these constants is bounded above by the budget balance condition.

Lemma 4. *A pair of collusive allocation and payment functions* $(p,t):\Theta\to\Sigma\times\mathbb{R}^n$ *in a grand mechanism* Γ *with menu* (Σ^*,c) *is ex ante budget balanced if and only if condition BB in Program C* given in Proposition 1 is satisfied.*

From Proposition 1 it is clear that the cartel will choose $\sum_{i=1}^n U_i(\underline{\theta})$ to be as high as possible. Hence the constraint BB must hold with equality at the cartel optimal solution: $E_\theta[\sum_{i=1}^n t_i(\theta)]=E_\theta[c(p(\theta))]$. This implies that the principal's expected revenue depends only on the allocation function p and is equal to $E_\theta[c(p(\theta))]$.

4.3 *Characterizing the solution to the cartel's problem*

In this section we provide a partial characterization of the solution to the problem of the cartel, which we later use to solve the problem of the principal. In particular we focus on the impact of the individual rationality constraints on the solution to the cartel's problem.

Consider an auxiliary problem of the cartel, where we drop the individual rationality constraints (described by *IR*) and replace the cartel's original welfare weight functions $\omega_1,\dots,\omega_n$ with new welfare weight functions $W_1,\dots,W_n$. The apparent similarity of the definition presented below to the concept of interim efficiency explains the name we give to the solution of this auxiliary problem of the cartel.[8]

Definition 2. An allocation function and agents' interim payoffs $(p,U_1,\dots,U_n)$ are *cartel interim efficient with respect to the menu* (Σ^*,c) *relative to the weights* $W_1,\dots,W_n$ if they solve

$$\text{Program CIE:} \qquad \max_{(p,U_1,\dots,U_n)} \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, d\, W_i(\theta_i) \quad \text{subject to}$$

$$F,\ IC, \text{ and } BB \text{ in Program C}^* \text{ given in Proposition 1.}$$

Consider the problem of a cartel facing a grand mechanism Γ with menu (Σ^*,c) and outside options $V_1,\dots,V_n$ (Program C*). In case the cartel's solution $(p,U_1,\dots,U_n)$ is such that none of the individual rationality constraints bind, it is clear that $(p,U_1,\dots,U_n)$ must be cartel interim efficient with respect to the menu (Σ^*,c) relative to the original weights $\omega_1,\dots,\omega_n$. The next result shows that if some of the individual rationality constraints bind, then $(p,U_1,\dots,U_n)$ must be cartel interim efficient with respect to the menu (Σ^*,c) *relative to some weights* $W_1,\dots,W_n$, which may differ from the original weights $\omega_1,\dots,\omega_n$.

Theorem 1. *Assume $(p,U_1,\dots,U_n)$ solves the problem of the cartel facing the grand mechanism Γ with menu (Σ^*,c) and outside options $V_1,\dots,V_n$ (Program C*). Then there exist weight functions $W_1,\dots,W_n$, which satisfy*

$$W_i(\overline{\theta})=1 \quad \textit{and} \quad \int_{\underline{\theta}}^{\overline{\theta}} d\, W_i(\theta)=1$$

for every i, such that $(p,U_1,\dots,U_n)$ is cartel interim efficient with respect to the menu (Σ^*,c) relative to the weights $W_1,\dots,W_n$ (i.e., solves Program CIE).

The intuition for this result is similar to the one present in the principal–agent problems when the agent has exogenous type-dependent outside options.[9] One could guess that due to the presence of the type-dependent individual rationality constraints the

[8] See Holmström and Myerson (1983).

[9] See for example Jullien (2000). See also Celik (2008) for a similar characterization of a collusive problem in a different setup.

solution to the original problem $(p, U_1, \dots, U_n)$ also solves the problem

$$\max_{(p,U_1,\dots,U_n)} \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, d\omega_i(\theta_i) + \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, d\Lambda_i(\theta_i) \quad \text{subject to } F, IC, \text{ and } BB,$$

where the auxiliary weight functions $\Lambda_1, \dots, \Lambda_n$ are such that $\Lambda_i(\theta_i)$ is the shadow value of a uniform relaxation of the individual rationality constraints for agent i's types that are weakly lower than θ_i. Thus the function Λ_i describes the pattern of binding individual rationality constraints for agent i and is naturally interpreted as the cumulative distribution function of the corresponding Lagrange multipliers in the cartel's problem. The proof confirms that this intuition is almost correct. There exist weight functions $W_1, \dots, W_n$ such that $W_i(\theta_i) = r_0 \omega_i(\theta_i) + \Lambda_i(\theta_i)$ for every i and θ_i, where r_0 is equal to either 0 or 1, such that the solution to the original problem $(p, U_1, \dots, U_n)$ also solves the problem

$$\max_{(p,U_1,\dots,U_n)} \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i(\theta_i)\, dW_i(\theta_i) \quad \text{subject to } F, IC, \text{ and } BB.$$

Moreover, it is possible to normalize $W_i(\overline{\theta}) = 1$ and $\int_{\underline{\theta}}^{\overline{\theta}} dW_i(\theta) = 1$ for every i. For the rest of the paper we restrict attention to weight functions $W_1, \dots, W_n$ normalized in this way.

The benefit of Theorem 1 is that it captures the effect of the individual rationality constraints on the solution to the cartel's problem. Recall that by offering a grand mechanism the principal determines two things: the menu (Σ^*, c) and the outside options $V_1, \dots, V_n$. The latter in turn implicitly determine the Lagrange multipliers corresponding to the individual rationality constraints, which force the cartel to maximize the agents' weighted expected payoffs according to the new welfare weights $W_1, \dots, W_n$. In the next section we study how much power the principal in fact has in determining these new welfare weights.

If we substitute the expressions for the agents' interim payoffs $U_1, \dots, U_n$ from condition *IC* in the problem describing the interim efficient allocation functions and agents' interim payoffs given in Program CIE then we obtain a definition of cartel interim efficient allocation functions.

Definition 3. An allocation function p is *cartel interim efficient with respect to the menu* (Σ^*, c) relative to the weights $W_1, \dots, W_n$ if it solves

Program CIE*: $$\max_p E\left[\sum_{i=1}^{n}\left(\theta_i + \frac{F(\theta_i) - W_i(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right] \quad \text{subject to}$$

*F**: $p(\theta) \in \Sigma^*$ for every $\theta \in \Theta$

*IC**: $E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$ is non-decreasing in θ_i for every i

*BB**: $E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right] \geq 0.$

A cartel interim efficient allocation function p with respect to the menu (Σ^*, c) relative to the weights $W_1, \ldots, W_n$ is *essentially unique* if any other cartel interim efficient allocation function p' with respect to the same menu and relative to the same weights is equivalent to p in the following sense:

$$\int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p'_i(\widetilde{\theta}_i, \theta_{-i})]\, d\widetilde{\theta}_i = \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})]\, d\widetilde{\theta}_i \quad \text{for every } \theta_i \text{ and every } i.$$

It is easy to show that if $(p, U_1, \ldots, U_n)$ is cartel interim efficient with respect to a given menu relative to some particular weights, then p is cartel interim efficient with respect to the same menu relative to the same weights. This follows from integration by parts and the fact that *BB* must hold with equality at the optimum. The following corollary follows directly from Theorem 1.

Corollary 1. *Assume* $(p, U_1, \ldots, U_n)$ *solves the problem of the cartel facing a grand mechanism* Γ *with menu* (Σ^*, c) *and outside options* $V_1, \ldots, V_n$ *(Program C*). Then there exist weight functions* $W_1, \ldots, W_n$ *such that p is cartel interim efficient with respect to the menu* (Σ^*, c) *relative to the weights* $W_1, \ldots, W_n$ *(i.e., solves Program CIE*).*

5. Solving the principal's problem

5.1 *Weakly collusion-proof mechanisms*

In this section we characterize grand mechanisms that leave no scope for collusion. When the cartel is facing such a mechanism, the best it can do is to replicate the allocations and interim expected payoffs that the agents can achieve by refusing to participate in the collusive mechanism and playing noncooperatively in the grand mechanism. These collusion-proof mechanisms are of interest because it turns out that in our setup there is no loss for the principal in optimizing over such mechanisms. The proof of this *collusion-proofness principle* is the same as the one in Laffont and Martimort (1997), and is therefore omitted.

The collusion-proof grand mechanisms introduced in this section are called weakly collusion-proof because of the assumption that the players do not update their beliefs following the rejection of the grand mechanism. In Section 6.3 we discuss uniquely collusion-proof grand mechanisms, which are robust against collusion in a stronger sense.

Fix a Bayesian Nash equilibrium in a given grand mechanism Γ following the rejection of a collusive mechanism assuming that the agents do not learn anything from the rejection of the side mechanism, and thus their beliefs about the types of the opponents coincide with their prior beliefs. The equilibrium strategies of the agents result in a *noncooperative allocation function* $p^N : \Theta \to \Sigma$ and a *noncooperative payment function* $t^N : \Theta \to \mathbb{R}^n$. Define the interim expected payoff of agent i of type θ_i from the given equilibrium by

$$U_i^N(\theta_i) = E_{\theta_{-i}}[\theta_i p_i^N(\theta_i, \theta_{-i}) - t_i^N(\theta_i, \theta_{-i})],$$

where $E_{\theta_{-i}}[\cdot]$ denotes the expectation with respect to the prior distribution over θ_{-i}. We say that a grand mechanism Γ *implements* p^N *and* $(U_1^N,\dots,U_n^N)$ if there exists an equilibrium in a given grand mechanism Γ (played by the agents under prior beliefs) resulting in an allocation function p^N and interim payoffs $(U_1^N,\dots,U_n^N)$.

Definition 4. A grand mechanism Γ with menu (Σ^*, c) that implements an allocation function p^N and interim payoffs $(U_1^N,\dots,U_n^N)$ is *weakly collusion-proof* if $(p^N, U_1^N,\dots,U_n^N)$ solves the cartel's problem facing the grand mechanism Γ with menu (Σ^*, c) and outside options $(U_1^N,\dots,U_n^N)$ (i.e., solves Program C*).

From Corollary 1, if an allocation function and interim expected payoffs $(p, U_1,\dots,U_n)$ solve the cartel's problem facing a grand mechanism Γ with menu (Σ^*, c) and some outside options (Program C*), then this allocation function p must be cartel interim efficient with respect to the menu (Σ^*, c) relative to some weights $W_1,\dots,W_n$ (i.e., it must solve Program CIE*). The next goal is to find out which cartel interim efficient allocations the principal can achieve in a collusion-proof way.

Theorem 2. *Assume* Σ^* *is a closed convex subset of* Σ *such that* $\mathbf{0} \in \Sigma^*$*, and* $c : \Sigma^* \to \mathbb{R}$ *is a convex function such that* $c(\mathbf{0}) \leq 0$*. Assume* p *is an essentially unique cartel interim efficient allocation with respect to the menu* (Σ^*, c) *relative to the weights* $W_1,\dots,W_n$ *(i.e., solves Program CIE*). Then there exists a weakly collusion-proof grand mechanism* Γ *with menu* (Σ^*, c) *that implements the allocation function* p*, irrespective of the cartel's welfare weights* $\omega_1,\dots,\omega_n$*.*

The grand mechanism Γ we construct is a direct mechanism, where in addition each agent is allowed to send a rejection message $\varnothing$, which triggers the null allocation and no payments. The vector of allocations for any given profile of reported types θ is given by $p(\theta)$. It is easy to construct incentive payments that assure truthful reporting by the agents, because the allocation function p satisfies the monotonicity constraints IC^*. Since p satisfies the constraint BB^*, it is possible to ensure that the interim payoffs of every type of every agent are nonnegative, and thus nobody has an incentive to send the rejection message $\varnothing$. Using Lemma 2 we can ensure that for any given profile of reported types θ the sum of the collected payments is equal to $c(p(\theta))$, which together with the condition F^* implies that the grand mechanism Γ has the menu (Σ^*, c).

The intuition that this grand mechanism is weakly collusion-proof is as follows. The cartel interim efficient allocation p with respect to the menu (Σ^*, c) relative to the weights $W_1,\dots,W_n$ solves Program CIE*, and thus maximizes the sum of the interim expected payoffs of the agents under the welfare weights $W_1,\dots,W_n$. Any alternative allocation that is not equivalent to p fails to be cartel interim efficient with respect to this menu and relative to these weights due to the essential uniqueness condition, and thus results in lower interim expected payoffs for some types of some agents. Hence any collusive proposal involving such an alternative allocation is vetoed by some types of some agents because they obtain higher payoffs by playing in the grand mechanism noncooperatively. This means that only collusive proposals involving allocations that are equivalent to p are not vetoed. However, it can be shown that such proposals result in exactly the same interim payoffs as in the noncooperative equilibrium of the grand mechanism specified above. Thus the cartel cannot do better than to simply replicate the noncooperative equilibrium of the grand mechanism.

Let us explain the role of essential uniqueness in the result. Assume the welfare weights $W_1, \ldots, W_n$ do not have full support. Then there might exist another cartel interim efficient allocation p' for which one can construct interim payoffs that are weakly higher than the interim payoffs in the grand mechanism Γ and are strictly higher for some types of some agents, which are outside the support of the welfare weights $W_1, \ldots, W_n$. In this case the cartel may be better off proposing a side mechanism resulting in an allocation p', which renders Γ not collusion-proof. The requirement of essential uniqueness precludes the existence of such an allocation p'.

Finally, let us note that the grand mechanism Γ constructed in the proof of Theorem 2 is weakly collusion-proof regardless of the cartel's welfare weight functions $\omega_1, \ldots, \omega_n$. This property of our weakly collusion-proof mechanisms is related to the concept of *robust collusion-proofness* introduced by Che and Kim (2006). However, weak collusion-proofness is somewhat weaker because it assumes passive beliefs following the rejection of the side mechanism, while robust collusion-proofness allows for a certain range of beliefs.

Corollary 1 to Theorem 1 and Theorem 2 together provide a characterization of the set of allocation functions that can be achieved by the principal in a weakly collusion-proof way. According to Corollary 1, if we fix a given menu, any such allocation must belong to the set of cartel interim efficient allocation functions. On the other hand, by Theorem 2, if we fix a given menu, any essentially unique cartel interim efficient allocation function can be achieved in a weakly collusion-proof way by the principal.

5.2 *Achieving the second best revenue*

In this section we describe the circumstances under which the principal incurs no revenue loss from collusion. The second best mechanism when there is no collusion is obtained by Myerson (1981). The allocation in the Myerson mechanism is given by

$$p_i(\theta_1, \ldots, \theta_n) = \begin{cases} 1 & \text{if } \theta_i > \max\{\max_{j \neq i} \theta_j, \theta^*\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for every } i \text{ and } (\theta_1, \ldots, \theta_n),$$

where θ^* is the cutoff type such that $\theta^* = (1 - F(\theta^*))/f(\theta^*)$ if such a value θ^* exists, and $\underline{\theta}$ otherwise. The expected revenue from the Myerson mechanism is

$$\Pi^* = \int_{\theta^*}^{\overline{\theta}} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) dF^n(\theta).$$

Theorem 3. *The Myerson revenue can be achieved by a weakly collusion-proof mechanism if and only if*

$$\theta^* + \frac{F(\theta^*)}{f(\theta^*)} \geq \frac{\Pi^*}{1 - F^n(\theta^*)}.$$

The proof of sufficiency is constructive. Denote $R^* = \Pi^*/(1-F^n(\theta^*))$. Consider the following linear and symmetric menu:

$$\Sigma^* = \Sigma \quad \text{and} \quad c(p) = R^* \sum_{i=1}^{n} p_i \text{ for every } p \in \Sigma.$$

When the cartel is facing a grand mechanism with such a menu, it can buy the good at the price R^* and allocate it to any agent. In case the cartel does not buy the good it can avoid any payments to the principal.

Consider the following symmetric weight functions, which assign all the weight to the highest type $\overline{\theta}$:

$$W_i(\theta_i) = \overline{W}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \in [\underline{\theta}, \overline{\theta}) \\ 1 & \text{if } \theta_i = \overline{\theta} \end{cases} \quad \text{for every } i.$$

Consider the Program CIE* describing the interim efficient allocations under the above weights and ignore the constraint BB^* for the moment. Notice that the virtual utility of every agent i is equal to $\theta_i + (F(\theta_i)/f(\theta_i))$, which is strictly monotonic by Condition 1. Thus for any given profile of types the cartel prefers to buy the good and allocate it to the agent with the highest type as long as this type exceeds a cutoff type $\widehat{\theta}$ such that $\widehat{\theta} + (F(\widehat{\theta})/f(\widehat{\theta})) = R^*$. By Condition 1, $\widehat{\theta} \leq \theta^*$ since $\theta^* + (F(\theta^*)/f(\theta^*)) \geq R^*$. The expected revenue of the principal is thus equal to $R^*(1-F^n(\widehat{\theta}))$, which is greater than Π^* (unless $\widehat{\theta} = \theta^*$). This is clearly impossible, and thus the constraint BB^* must bind and forces the cartel to implement precisely the cutoff type θ^*:

$$E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)p_i(\theta)\right] - R^* E\left[\sum_{i=1}^{n} p_i(\theta)\right]$$
$$= \int_{\theta^*}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) dF^n(\theta) - R^*(1-F^n(\theta^*)) = \Pi^* - \Pi^* = 0.$$

In case $\theta^* + (F(\theta^*)/f(\theta^*)) < R^*$ the Myerson revenue cannot be achieved. The cartel's optimal cutoff $\widehat{\theta}$ is now greater than θ^*, which can be shown not to violate BB^*. This results in the expected revenue $R^*(1-F^n(\widehat{\theta}))$, which is smaller than Π^*. The proof of necessity demonstrates that in this case no weakly collusion-proof grand mechanism can achieve the Myerson revenue.

The condition for achieving the Myerson revenue present in the statement of Theorem 3 is more likely to hold when the number of agents is small. This is due to the fact that θ^*, and thus $\theta^* + (F(\theta^*)/f(\theta^*))$, is independent of n, while the right-hand side is increasing in n. Also notice that this condition fails if the Myerson mechanism involves no exclusion: $\theta^* = \underline{\theta}$. In these cases the left-hand side is just equal to $\underline{\theta}$, while the right-hand side is strictly greater than $\underline{\theta}$.

Example 1. Let $n = 2$ and let the types be distributed independently and uniformly on $[\underline{\theta}, \underline{\theta}+1]$ where $\underline{\theta} \geq 0$. By Theorem 3 the Myerson revenue is achieved when $\underline{\theta} \leq \frac{1}{2}(7-\sqrt{33}) \approx 0.628$.[10] ◊

5.3 *The optimal linear symmetric mechanism when the second best cannot be achieved*

The next goal is to study the problem of the principal when the Myerson revenue cannot be achieved. This turns out to be a hard problem, so in this section we restrict attention to the class of grand mechanisms with linear and symmetric menus, while in Section 6.1 we evaluate the scope for grand mechanisms with nonlinear and asymmetric menus.

Definition 5. A grand mechanism Γ has a *linear symmetric menu with a price R* if

$$\Sigma^* = \Sigma \quad \text{and} \quad c(p) = R\sum_{i=1}^{n} p_i \text{ for every } p \in \Sigma.$$

This class of mechanisms is of interest in its own right. In Section 4.1 we argue that many standard mechanisms, including first- and second-price auctions with a common reservation price, have linear symmetric menus. Moreover, in Section 5.2 we have seen that one particular member of this class of mechanisms successfully achieves the Myerson revenue.

Lemma 5. *Assume the principal is restricted to offer grand mechanisms with linear symmetric menus with a given price R. The highest revenue is achieved by a weakly collusion-proof grand mechanism that implements a cartel interim efficient allocation with respect to this menu relative to the weight functions*

$$W_i(\theta_i) = \overline{W}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \in [\underline{\theta}, \overline{\theta}) \\ 1 & \text{if } \theta_i = \overline{\theta} \end{cases} \quad \text{for every } i.$$

The principal determines the virtual utilities $\theta_i + ((F(\theta_i) - W_i(\theta_i))/f(\theta_i))$ for Program CIE* describing the cartel interim efficient allocations by choosing the weight functions $W_1, \ldots, W_n$. The higher the virtual utility, the more often the cartel buys the good, and thus the higher is the principal's revenue. The highest possible virtual utility is achieved by the weight function $\overline{W}$ and is equal to $\theta_i + (F(\theta_i)/f(\theta_i))$. Thus the preferences of the principal are best aligned with the preferences of the agents with the highest valuations. The complication comes from the need to satisfy the constraint BB^*. It may happen that the most preferred allocation for a given virtual utility violates the constraint. The proof shows that the principal does best by choosing the weight functions $\overline{W}, \ldots, \overline{W}$ even when the cartel has to satisfy the constraint BB^*.

[10] The calculations are available in a supplementary file on the journal website, http://econtheory.org/supp/243/supplement.pdf.

THEOREM 4. *Assume $\theta^* + (F(\theta^*)/f(\theta^*)) < \Pi^*/(1 - F^n(\theta^*))$. Assume the principal is restricted to offer grand mechanisms with linear symmetric menus. The highest revenue is achieved by a weakly collusion-proof grand mechanism that implements an interim efficient allocation with respect to the menu with the price R^{**} relative to the weight functions $\overline{W},\dots,\overline{W}$ where*

$$R^{**} = \theta^{**} + \frac{F(\theta^{**})}{f(\theta^{**})} \quad \text{and} \quad \theta^{**} = \arg\max_{\theta\in[\underline{\theta},\theta']} \left(\theta + \frac{F(\theta)}{f(\theta)}\right)(1 - F^n(\theta)),$$

and where $\theta' \in (\underline{\theta}, \overline{\theta})$ is the unique value such that

$$\int_{\theta}^{\overline{\theta}} \left(\widetilde{\theta} - \frac{1 - F(\widetilde{\theta})}{f(\widetilde{\theta})} - \left(\theta + \frac{F(\theta)}{f(\theta)}\right)\right) dF^n(\widetilde{\theta}) \gtreqqless 0 \quad \text{when} \quad \theta \lesseqqgtr \theta'.$$

Due to the result established in Lemma 5, the problem of the optimal choice of the price R is equivalent to the problem of a monopolist dealing with a single buyer whose valuation is given by $\theta + (F(\theta)/f(\theta))$ when his type is θ, and the types are distributed according to the cumulative distribution function F^n. This buyer operates under an additional constraint BB^*. In the Appendix we establish a technical result that this constraint binds only when the price R is greater than $\theta' + (F(\theta')/f(\theta'))$, and in the proof of the theorem we show that it is never profitable to charge such a high price.

6. Extensions and discussion

6.1 *On asymmetric and nonlinear mechanisms*

In this section we show that when the Myerson revenue cannot be achieved, the principal may benefit from grand mechanisms with asymmetric menus (Example 2) as well as nonlinear menus (Example 3). We give only the main ideas here.[11]

EXAMPLE 2. Let $n = 2$ and let the types be distributed independently and uniformly on $[10, 11]$. There exists a grand mechanism with an asymmetric menu that achieves a higher revenue than the best grand mechanism with a linear and symmetric menu. ◊

The best grand mechanism with a linear and symmetric menu (Theorem 4) implements a cartel interim efficient allocation relative to the weight functions $W_1 = \overline{W}$ and $W_2 = \overline{W}$ with respect to the menu $c(p_1, p_2) = Rp_1 + Rp_2$, where R is approximately 10.194. The principal's revenue is approximately 10.098.

Consider an alternative cartel interim efficient allocation with respect to the asymmetric menu $c(p_1, p_2) = (R + \varepsilon)p_1 + Rp_2$, where R is the same as above and $\varepsilon = 0.05$, relative to the weight functions $W_1 = \overline{W}$ and $W_2 = \widetilde{W}$ where

$$\widetilde{W}(\theta_2) = \begin{cases} 0 & \text{if } \theta_2 \in [10, 10.75) \\ 1 & \text{if } \theta_2 \in [10.75, 11]. \end{cases}$$

[11] The calculations are available in a supplementary file on the journal website, http://econtheory.org/supp/243/supplement.pdf.

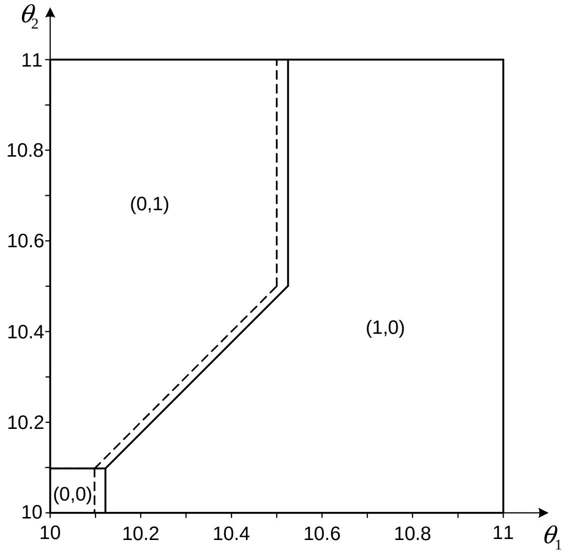

FIGURE 1. The allocation in Example 2.

The allocation is given in Figure 1. No agent gets the good and there is no payment to the principal when the realization of the types (θ_1, θ_2) belongs to the subset denoted by $(0,0)$. Agent 1 gets the good and the principal receives $R+\varepsilon$ when (θ_1, θ_2) belongs to the subset denoted by $(1,0)$. Agent 2 gets the good and the principal receives R when (θ_1, θ_2) belongs to the subset denoted by $(0,1)$. This allocation is essentially unique and thus by Theorem 2 there exists a grand mechanism that implements it in a weakly collusion-proof way. The revenue is approximately 10.103, which is higher than the revenue from the best grand mechanism with a linear symmetric menu.

The intuition for this result is as follows. If we just introduced the new weight functions as above and left the menu unchanged ($\varepsilon = 0$), the cartel would still buy the good from the principal in the same states of the world as before, and only the allocation of the good between the agents would change. The new weight functions result in a higher virtual utility for the first agent, and thus he now gets the good more often. This allocation is denoted by the dashed lines in Figure 1.

Now consider increasing the price of the good for the first agent. Since the price R is optimal in the symmetric mechanism, it optimally trades off the considerations of obtaining higher revenue from the inframarginal types and losing revenue from the marginal types. The mass of the inframarginal types of the first agent under the new allocation is higher. Thus it pays the principal to charge the first agent a higher price. The final allocation is denoted by the solid lines in Figure 1.

Let us note that the constructed grand mechanism with an asymmetric linear menu is not the optimal one. It just performs better than the best grand mechanism with a linear symmetric menu. Also we do not know whether the fully optimal grand mechanism has a asymmetric menu since we have kept the linearity assumption intact.

Next we show how the principal may benefit from offering grand mechanisms with nonlinear menus. The example is similar in spirit to the one-principal and one-agent model in Thanassoulis (2004), where the agent has multidimensional private information.

Example 3. Let $n=2$ and let the types be distributed independently and uniformly on $[10,11]$. There exists a grand mechanism with a nonlinear menu that achieves a higher revenue than the best grand mechanism with a linear and symmetric menu. ◊

Consider the grand mechanism with an asymmetric menu from Example 2. In addition, offer the cartel an option to buy a lottery at a price R under which the good is equally likely to be allocated to either agent. Thus $c(0,0)=0$, $c(1,0)=R+\varepsilon$, and $c(0,1)=c(\frac{1}{2},\frac{1}{2})=R$. The cost of any other allocation is determined by the lower envelope (vex) over these four points.

The cartel interim efficient allocation is given in Figure 2. No agent gets the good and there is no payment to the principal when the realization of the types (θ_1,θ_2) belongs to the subset denoted by $(0,0)$. Agent 1 gets the good and the principal receives $R+\varepsilon$ when (θ_1,θ_2) belongs to the subset denoted by $(1,0)$. Agent 2 gets the good and the principal receives R when (θ_1,θ_2) belongs to the subset denoted by $(0,1)$. The allocation is determined by a fair lottery and the principal receives R when (θ_1,θ_2) belongs to the subset denoted by $(\frac{1}{2},\frac{1}{2})$. This allocation is essentially unique and thus by Theorem 2 there exists a grand mechanism that implements it in a weakly collusion-proof way. The revenue is approximately 10.105, which is higher than the revenue from the grand mechanism with a linear asymmetric menu in Example 2, and thus is also higher than the revenue from the best grand mechanism with a linear symmetric menu. The intuition behind the result is simple. The allocation prior to the introduction of the lottery is denoted by the dashed lines in Figure 2. Offering a $(\frac{1}{2},\frac{1}{2})$ lottery at the price R has two effects. In the instances when the virtual utilities of the agents are close to each other, the cartel switches to purchasing the lottery since it is not that important who gets the good. This results in a revenue loss for the principal. But there is also a positive effect in that the cartel now buys in some states of the world where previously it did not. In our example the second effect outweighs the first. The final allocation is denoted by the solid lines in Figure 2.

6.2 *Reallocations*

Recall that throughout the paper we assume that reallocations of the good between the agents are not allowed. We now discuss what happens when the principal has no control over reallocations.

In this case we must allow for reallocations of the good between the agents as part of a side mechanism S. Define a reallocation function $z : M\times\Theta\to\mathbb{R}^n$ which associates with each message profile reported to the grand mechanism and with each profile of the reported types a vector of reallocation adjustments. The reallocation function must be ex post feasible and is assumed to be ex post balanced: $\tilde{p}(m)+z(m,\theta)\in\Sigma$ and

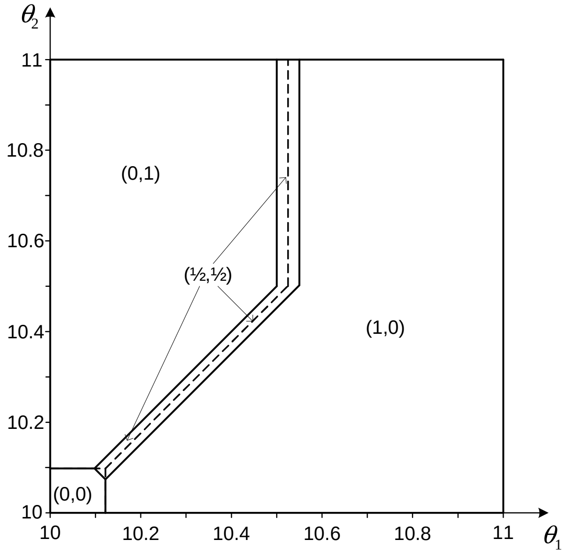

FIGURE 2. The allocation in Example 3.

$\sum_{i=1}^n z_i(m,\theta)=0$ for every $m \in M$ and $\theta \in \Theta$.[12] At the final node of the game ($t=7$) the cartel performs the reallocation adjustments $z(m,\widehat{\theta})$.

DEFINITION 6. A pair of collusive allocation and payment functions $(p,t):\Theta \to \Sigma \times \mathbb{R}^n$ is *feasible and budget balanced* in the grand mechanism $\Gamma = \langle M, \widetilde{p}, \widetilde{t}\rangle$ if there exists a side mechanism $S=\langle \Theta, \phi, y, z\rangle$ such that

(i) $p_i(\theta) = E_{\phi(\theta)}[\widetilde{p}_i(m)+z_i(m,\theta)]$ and $t_i(\theta) = E_{\phi(\theta)}[\widetilde{t}_i(m)] - y_i(\theta)$ for every i and $\theta \in \Theta$, where $E_{\phi(\theta)}[\cdot]$ denotes the expectation according to $\phi(\theta)$ over M

(ii) $\sum_{i=1}^n y_i(\theta) \le 0$ and $\sum_{i=1}^n z_i(m,\theta)=0$ for every $m \in M$ and $\theta \in \Theta$.

For any given grand mechanism $\Gamma = \langle M, \widetilde{p}, \widetilde{t}\rangle$ we construct a menu that takes into account the cartel's possibilities for reallocations of the good. Define the set of allocations that the cartel can achieve by sending various message profiles in the grand mechanism and reallocating the good between the agents:

$$\Sigma_1^r = \{p \in \Sigma \mid \text{there exists } m \in M \text{ such that } \textstyle\sum_{i=1}^n p_i = \sum_{i=1}^n \widetilde{p}_i(m)\}.$$

Define Σ^r to be a convex hull of Σ_1^r:

$$\Sigma^r = \text{co}(\Sigma_1^r).$$

Notice that Σ^r has the following form: $\Sigma^r = \{p \in \Sigma \mid \sum_{i=1}^n p_i \le \overline{p}\}$ where $\overline{p} \in [0,1]$.

[12]It is straightforward to incorporate free disposal of the good in the model of collusion, but it would make the notation more cumbersome without adding much value.

Define the function $c_1^r : \Sigma_1^r \to \mathbb{R}$ to be the lowest bound for the aggregate cost for the cartel at which any given allocation p can be attained by purchasing it from the principal and possibly reallocating it between the agents:

$$c_1^r(p) = \inf_{\{m:\ \sum_{i=1}^n \widetilde{p}_i(m) = \sum_{i=1}^n p_i\}} \sum_{i=1}^n \widetilde{t}_i(m) \quad \text{for every } p \in \Sigma_1^r.$$

Define the function $c^r : \Sigma^r \to \mathbb{R}$ to be the greatest convex function that is pointwise weakly lower than c_1^r:

$$c^r(p) = \operatorname{vex}(c_1^r(p)) \quad \text{for every } p \in \Sigma^r.$$

Notice that for any p both $c_1^r(p)$ and $c^r(p)$ depend only on the sum of probabilities $\sum_{i=1}^n p_i$. Also we must have $c^r(\mathbf{0}) \leq 0$ since the cartel can always achieve the null allocation and avoid any payments by sending the rejection message $\varnothing$ to the grand mechanism on behalf of every agent.

Next we present an analog of Lemma 1 from Section 4.1 to obtain a tractable characterization for the feasibility and budget balance constraints in this model.

Lemma 6. *Consider a grand mechanism Γ with menu (Σ^r, c^r). A pair of collusive allocation and payment functions $(p, t) : \Theta \to \Sigma \times \mathbb{R}^n$ is feasible and budget balanced in Γ if and only if*

(i) $p(\theta) \in \Sigma^r$ for every $\theta \in \Theta$

(ii) $t(\theta)$ is such that $\sum_{i=1}^n t_i(\theta) \geq c^r(p(\theta))$ for every $\theta \in \Theta$.

It is easy to see that under the reallocation scenario the results of Sections 4.2 and 4.3 hold if we replace the menu (Σ^*, c) with the menu (Σ^r, c^r). For Theorem 2 of Section 5.1 to be valid we need to replace the first sentence with: "Assume $\Sigma^r = \{p \in \Sigma \mid \sum_{i=1}^n p_i \leq \overline{p}\}$, where $\overline{p} \in [0,1]$, and $c^r : \Sigma^r \to \mathbb{R}$ is a convex function such that $c(\mathbf{0}) \leq 0$ and $c^r(p) = c^r(p')$ whenever $\sum_{i=1}^n p_i = \sum_{i=1}^n p_i'$." Notice that linear symmetric menus satisfy these conditions, and thus all the results of Sections 5.2 and 5.3 hold.[13] In particular the optimal linear symmetric mechanisms in Theorems 3 and 4 remain weakly collusion-proof even if the cartel has the ability to reallocate the good between the agents. For the same reason Theorem 5 presented in the next section is also valid under the reallocation scenario.

The examples of mechanisms with asymmetric and nonlinear menus presented in Section 6.1 do not satisfy the above restrictions on the menus, and thus are not weakly collusion-proof under the reallocation scenario.

[13]We conjecture that under the reallocation scenario the optimal mechanisms always have linear symmetric menus.

6.3 *Multiple equilibria and uniquely collusion-proof mechanisms*

One weakness of our optimal mechanisms is the focus on equilibria in which the agents do not update their beliefs following the rejection of the collusive mechanism. Although this approach is fully legitimate from the perspective of mechanism design theory, one might argue that there may exist other, perhaps more plausible equilibria where the agents learn something about their opponents upon observing the rejection of the proposed collusive scheme.

Cramton and Palfrey (1995) propose the concept of *ratifiable equilibrium* in which the agents update their beliefs following the rejection of the proposal in a plausible way. Depending on the game, the outside options arising from ratifiable equilibria may be higher or lower than the outside options generated by equilibria with passive updating, and may thus lead to the tightening or relaxation of the individual rationality constraints in the cartel's problem.

This approach can be applied when there is a single collusive proposal and a well-defined game to be played in case of rejection (and one can solve for asymmetric equilibria). However, in our setup the prospects of using this approach are quite grim since the game to be played by the agents is itself a choice variable of the principal. Instead, later in this section we use tools from implementation theory to address the issue of multiple equilibria.

Another interesting approach is to consider an equilibrium where the beliefs used by the rest of the agents about the type of the agent who rejects a collusive proposal are chosen to maximize the interim payoffs of this agent, and thus lead to the maximal tightening of the individual rationality constraints in the cartel's problem.[14] Such an equilibrium is likely to be the most preferred one by the principal, and, following a standard approach in the mechanism design theory, the principal can choose this equilibrium to be played following the rejection of the collusive proposal.[15]

We feel that such an approach stacks the deck in favor of the principal a little too much. For example, consider the second-price auction with a reserve price that implements the optimal mechanism and achieves the Myerson revenue when the agents do not collude. Suppose that after observing a rejection of the collusive proposal by one of the agents the rest of the agents believe that the deviator's type is above the reserve price. Then there exists an equilibrium where the deviator submits a bid above the highest possible type and the rest of the agents stay out, and thus the deviator receives the good for sure at the reserve price. It is not difficult to show that the problem of the cartel has no solution when the outside options are determined by the above equilibrium. We are then forced to make a counterintuitive conclusion that the second-price auction with a reserve price achieves the Myerson revenue because the collusion can be blockaded by an appropriate choice of beliefs and equilibrium following the rejection of the collusive proposal![16]

[14]This approach is the polar opposite to the one pursued by Caillaud and Jéhiel (1998), who consider a *minmax equilibrium* in which the beliefs are chosen to minimize the interim payoffs of the agent who rejects the collusive proposal.

[15]We thank one of the referees for pointing this out.

[16]In this example agents use weakly dominated strategies in equilibrium. It is possible to construct other

There is another issue with the analysis performed so far. Notice that the individual rationality constraints in the statement of the cartel's problem require that every type of every agent must be willing to participate in the cartel in order for collusion to take place. For example, this requirement of "full participation" rules out scenarios where the cartel is formed only if all the agents have high types, and otherwise the agents play in the grand mechanism noncooperatively.[17] To the best of our knowledge the "full participation" requirement is present in all studies of collusion under asymmetric information, and thus a discussion of the generality of this setup (or lack thereof) is in place.

For any given "incomplete participation" of this sort it is always possible to construct a side mechanism that induces full participation of every type of every agent and achieves the same allocation and payment functions as those resulting from incomplete collusion. This can be done by constructing a message reporting function that replicates the noncooperative play in case the cartel is not formed, and using null side payments. It is easy to see that the new collusive allocation and payment functions are incentive compatible (by the incentive compatibility of the original equilibrium). So far the "full participation" requirement proves to be without loss of generality. The problem may arise when we turn attention to the individual rationality constraints. Indeed, Celik and Peters (2008) present an intriguing example where the outside options in case of "incomplete participation" differ from the outside options that can arise from any equilibrium satisfying the "full participation" requirement. Verifying whether the "full participation" requirement involves any loss of generality in our model does not seem feasible because of the technical problems mentioned at the beginning of the section. Instead we use a result from implementation theory to address all of the above issues.

Palfrey and Srivastava (1993) describe the allocation functions that can be uniquely implemented in quasi-linear private value environments in a way that is robust to preplay communication and interim renegotiation. They show that the set of the allocation functions that are implementable in this robust way coincides with the set of interim efficient allocations in a given environment. First, any interim inefficient allocation can be improved upon by some alternative allocation function and is thus not implementable. On the other hand, for any interim efficient allocation one can construct a direct mechanism that has a truthful equilibrium resulting in this allocation. The difficult part is to ensure that there are no other equilibria resulting in some alternative allocation. Notice that, since the original allocation is interim efficient, any alternative allocation must necessarily make some types of some agents worse off relative to the original allocation. Palfrey and Srivastava (1993) augment the direct mechanism by giving to each agent a set of messages that trigger specifically designed individual allocations and payments resulting in the same interim payoffs (minus $\varepsilon > 0$ chosen by the agent) as the desired allocation function. In this way all undesirable equilibria of the direct mechanism are vetoed, and the standard techniques of implementation theory are used to make sure that no other equilibria are created.

We use an adaptation of the Palfrey and Srivastava (1993) approach and introduce a notion of uniquely collusion-proof mechanisms.

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examples without this undesirable feature, but doing so requires a more elaborate argument.

[17] We thank one of the referees for pointing this out.

DEFINITION 7. A grand mechanism Γ with menu (Σ^*, c) that implements an allocation function p^N and interim payoffs $(U_1^N, \ldots, U_n^N)$ is *uniquely collusion-proof* if every equilibrium of the game results in the allocation function p^N and interim payoffs $(U_1^N, \ldots, U_n^N)$.

Notice that unique collusion-proofness imposes a requirement that must be satisfied by every equilibrium of the game. These equilibria may entail either acceptance or rejection of the side mechanism on the equilibrium path. In case of equilibrium acceptance of the side mechanism the beliefs following rejection of the side mechanism are not restricted in any way.

The optimal mechanisms from Theorems 3 and 4 can be implemented in a uniquely collusion-proof way. Both of those mechanisms are implementing a cartel interim efficient allocation with respect to a linear symmetric menu with a specifically chosen price relative to weight functions that assign all the welfare weight to the highest type of each agent. We take a weakly collusion-proof grand mechanism from the proof of Theorem 2 and extend the set of messages available to each agent with an option to "buy the good now" at a price that provides the highest type of each agent with the same interim payoff (minus $\varepsilon > 0$ chosen by the agent) as the desired allocation function. These extra options ensure the unique collusion-proofness of the grand mechanism.

THEOREM 5. *Assume p is a cartel interim efficient allocation with respect to the linear symmetric menu with a price R relative to the weight functions $\overline{W}, \ldots, \overline{W}$ where*

$$R = \begin{cases} \Pi^*/(1-F^n(\theta^*)) & \text{if } \theta^* + (F(\theta^*)/f(\theta^*)) \geq \Pi^*/(1-F^n(\theta^*)) \\ R^{**} & \text{if } \theta^* + (F(\theta^*)/f(\theta^*)) < \Pi^*/(1-F^n(\theta^*)) \end{cases}$$

and R^{**} is defined as in Theorem 4. Then there exists a uniquely collusion-proof grand mechanism Γ with this menu that implements the allocation function p.

6.4 *Related literature*

Our model builds on the methodology developed by Laffont and Martimort (1997, 2000). They show that the agents' asymmetric information may prevent them from colluding efficiently. The principal is then able to exploit the resulting transaction costs of the collusive arrangement to her benefit. In their model the cartel maximizes the agents' aggregate sum of ex ante payoffs, the participation decisions are made noncooperatively, and each agent has only two possible types. Under the independence of the private information, Laffont and Martimort (1997) show that the principal can prevent collusion at no cost by allowing the collusion to operate efficiently, i.e., the principal finds it optimal to implement an interim efficient allocation relative to the cartel's welfare weights.

Che and Kim (2006) significantly generalize the results of Laffont and Martimort (1997, 2000). In particular they show that when the private information is independently distributed the principal can always fight off collusion at no cost using a *robustly collusion-proof* grand mechanism that works for any cartel welfare weight functions and allows for a certain range of beliefs following the rejection of the collusive mechanism.

Their construction exploits the fact that the agents make the participation decisions noncooperatively. The principal's grand mechanism has the property that the sum of the agents' payments is always equal to the expected Myerson revenue. This implies that the only allocation the cartel can implement without violating the constraints is the Myerson allocation. In our setup such a construction does not work because the agents can collude on their participation decisions, and thus the cartel can always achieve the null allocation at no cost.[18]

Dequiedt (2007) (see also Quesada 2005) studies a two-type model in an auction environment with the possibility of collusion on participation decisions. Moreover the cartel can commit to punishment strategies in case an agent refuses to join the collusive agreement. In this model there are also cases when the principal can achieve the Myerson revenue. This happens when it is optimal to allocate the good only to the high types. Dequiedt (2007) finds that the principal does not gain from introducing inefficiency in collusion. In contrast we show that the principal in general benefits from implementing the interim efficient allocations relative to weights that differ from the cartel's original welfare weights. We are not certain whether this difference is due entirely to the fact that Dequiedt (2007) considers two-type distributions of the agents' private information, or whether the assumption that the cartel can commit to punishment strategies in case an agent refuses to join the collusive agreement also plays a role.

In Mookherjee and Tsumagari (2004) the private information is distributed continuously and collusion on participation decisions is allowed. They do not solve for the optimal mechanism because their primary interest is to study the relative performance of different organizational arrangements: centralization with collusion, delegation, and intermediation. In terms of their model we provide a solution to the design of the optimal grand mechanism in the case of centralization with collusion.

In complementary work, Che and Kim (2007) independently obtain many of the results of this paper. They use a different approach and focus on the scenario in which the principal cannot prohibit reallocations of the good. Che and Kim (2007) have an analogue of our Theorem 3, which provides a weakly collusion-proof mechanism that achieves the Myerson revenue when it is possible to do so, and extend this result to the case of asymmetric agents. They also have an analogue of our Theorem 4, which provides the best linear symmetric weakly collusion-proof mechanism when the Myerson revenue cannot be achieved. For the case of symmetric bidders, Che and Kim (2007) are also able to handle the case in which only a subset of the agents participate in the cartel, which is not covered in this paper. It turns out that in this case there always exists a weakly collusion-proof mechanism that achieves the Myerson revenue.

Che and Kim (2007) manage to obtain their results without providing a complete characterization of the set of allocations that can be achieved by weakly collusion-proof mechanisms, as presented in our Theorems 1 and 2. We believe that this characterization has stand-alone value as demonstrated in Section 6.1, where Theorem 2 is used to provide examples of weakly collusion-proof mechanisms that outperform the best linear symmetric mechanisms when reallocations can be prohibited.

[18] See Section 2 for an example.

7. Conclusion

While our research addresses the problem of optimal mechanism design when the participating agents collude both on the way they play the mechanism and on their participation decisions, the methodology developed can be applied to a much broader class of environments than the auction environment considered in this paper. It is also possible to treat cases of heterogeneous agents and to allow partial cartels, as demonstrated by Che and Kim (2007).

There remain a number of open questions. For example, it is important to study the robustness of the results to alternative assumptions about the formation and operation of the cartel. It also would be interesting to study the case in which the principal is uncertain about the number of agents who may participate in her mechanism, because the current design of the optimal mechanism relies on the assumption that the principal has this information.

In Section 6.3 we discuss another set of pressing issues that have to do with the multiplicity of equilibria of the overall game and with exploring the possibility of cartels with "incomplete participation." In this paper we suggest a notion of uniquely collusion-proof mechanisms as a way of addressing these issues. However, our proposed fix may be of limited interest for practical applications since it utilizes theoretical constructs from implementation theory that are often perceived as being impracticable.

Another interesting extension is to study the effect of relaxing the budget balance constraint on the side payments inside the cartel. This question becomes relevant when the cartel activity extends beyond a given auction, and thus the cartel could bring in extra funds to subsidize collusion.[19] In this scenario the collusion-proofness principle is no longer valid: it is possible that a principal may welcome collusion because of the possibility of capturing some of the cartel's extra funds. The effect on the agents' ability to collude seems to depend on the specifics of the formal model and awaits future research.

Appendix

A. Proofs for Section 4

Proof of Lemma 1. Assume (p,t) is feasible and budget balanced in Γ. Take any $\theta\in\Theta$. Since $p(\theta)=E_{\phi(\theta)}[\tilde{p}(m)]$ and $\tilde{p}(m)\in\Sigma_1$ for every $m\in M$, we have $p(\theta)\in \text{co}(\Sigma_1)=\Sigma^*$. By definition of the function c we must have

$$\sum_{i=1}^{n}\tilde{t}_i(m)\geq c(\tilde{p}(m)) \quad \text{for every } m\in M.$$

Hence

$$E_{\phi(\theta)}\left[\sum_{i=1}^{n}\tilde{t}_i(m)\right]\geq E_{\phi(\theta)}[c(\tilde{p}(m))]\geq c(E_{\phi(\theta)}[\tilde{p}(m)])=c(p(\theta)),$$

[19] We thank one of the referees for suggesting this extension.

where the second inequality follows from convexity of c. Using the budget balance condition,

$$0 \geq \sum_{i=1}^{n} y_i(\theta) = E_{\phi(\theta)}\left[\sum_{i=1}^{n} \tilde{t}_i(m)\right] - \sum_{i=1}^{n} t_i(\theta) \geq c(p(\theta)) - \sum_{i=1}^{n} t_i(\theta).$$

Hence $p(\theta)$ and $t(\theta)$ satisfy (i) and (ii).

Take any $\theta \in \Theta$ and consider a vector of allocations $p(\theta)$ and a vector of payments $t(\theta)$ that satisfy (i) and (ii).

Define the set $D = \{\tilde{p}(m), \sum_{i=1}^{n} \tilde{t}_i(m) \mid m \in M\}$, and let $D^* = \text{co}(D)$. Notice that $(p(\theta), c(p(\theta))) \in D^*$ by definition of c and by Condition 2. By Carathéodory's theorem[20] any point in the set D^* can be represented as a convex combination of at most $n+2$ points in the set D. Thus there exist $d^1, \ldots, d^K$ in the set D (where $K \leq n+2$) and $\sigma_1, \ldots, \sigma_K$ from $\mathbb{R}_+^K$ such that $\sum_{k=1}^{K} \sigma_k = 1$ and

$$(p(\theta), c(p(\theta))) = \sum_{k=1}^{K} \sigma_k d^k.$$

By the definition of the set D there exist messages $m^1, \ldots, m^K$ such that

$$\left(\tilde{p}(m^k), \sum_{i=1}^{n} \tilde{t}_i(m^k)\right) = d^k \quad \text{for every } k.$$

Hence the cartel can achieve an allocation $p(\theta)$ and a vector of payments $\sum_{k=1}^{K} \sigma_k \tilde{t}(m^k)$ by sending the messages $m^1, \ldots, m^K$ with the probabilities $\sigma_1, \ldots, \sigma_K$. To achieve the desired vector of payments $t(\theta)$ we need the following vector of side payments:

$$y(\theta) = \sum_{k=1}^{K} \sigma_k \tilde{t}(m^k) - t(\theta).$$

Finally, we need to verify that $y(\theta)$ is budget balanced:

$$\sum_{i=1}^{n} y_i(\theta) = \sum_{i=1}^{n} \sum_{k=1}^{K} \sigma_k \tilde{t}_i(m^k) - \sum_{i=1}^{n} t_i(\theta) = c(p(\theta)) - \sum_{i=1}^{n} t_i(\theta) \leq 0,$$

where the inequality follows from (ii). Hence $(p(\theta), t(\theta))$ is feasible and budget balanced in Γ. □

Proof of Lemma 2. Denote by $\delta(\theta)$ the surplus/deficit of the cartel in the state $\theta \in \Theta$:

$$\delta(\theta) = \sum_{i=1}^{n} t_i(\theta) - c(p(\theta)).$$

[20]See for example Section 17 in Rockafellar (1970).

Construct a new payment function as follows:

$$t_i'(\theta) = t_i(\theta) + (1/n)\big(E_{\widetilde{\theta}}[\delta(\widetilde{\theta})] - E_{\widetilde{\theta}_{-(i+1)}}[\delta(\theta_{i+1}, \widetilde{\theta}_{-(i+1)})] + E_{\widetilde{\theta}_{-i}}[\delta(\theta_i, \widetilde{\theta}_{-i})] - \delta(\theta)\big),$$

for every i and every $\theta \in \Theta$, where by $n+1$ we mean 1.

Notice that $E_{\widetilde{\theta}_{-i}}[t_i'(\theta_i, \widetilde{\theta}_{-i})] = E_{\widetilde{\theta}_{-i}}[t_i(\theta_i, \widetilde{\theta}_{-i})]$, and thus the interim expected payments are unchanged. Finally, verify that the new payment function is budget balanced:

$$\sum_{i=1}^{n} t_i'(\theta) = \sum_{i=1}^{n} t_i(\theta) + E_{\widetilde{\theta}}[\delta(\widetilde{\theta})] - \delta(\theta) = c(p(\theta)) + E_{\widetilde{\theta}}[\delta(\widetilde{\theta})] \leq c(p(\theta)),$$

where the last equality follows from the fact that (p,t) is ex ante budget balanced. □

Proof of Lemma 4. The result follows from the accounting identities:

$$\begin{aligned} E\Big[\sum_{i=1}^{n} \theta_i p_i(\theta) - c(p(\theta))\Big] &= E\Big[\sum_{i=1}^{n} (\theta_i p_i(\theta) - t_i(\theta))\Big] + E\Big[\sum_{i=1}^{n} t_i(\theta) - c(p(\theta))\Big] \\ &= E\Big[\sum_{i=1}^{n} U_i(\theta_i)\Big] + E\Big[\sum_{i=1}^{n} t_i(\theta) - c(p(\theta))\Big] \\ &= \sum_{i=1}^{n} U_i(\underline{\theta}) + E\Big[\sum_{i=1}^{n} \frac{1-F(\theta_i)}{f(\theta_i)} p_i(\theta)\Big] + E\Big[\sum_{i=1}^{n} t_i(\theta) - c(p(\theta))\Big], \end{aligned}$$

where the last equality follows from Lemma 3 and integration by parts. Thus the sum of the payoffs of the lowest types is

$$\sum_{i=1}^{n} U_i(\underline{\theta}) = E\Big[\sum_{i=1}^{n} \Big(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\Big) p_i(\theta) - c(p(\theta))\Big] - E\Big[\sum_{i=1}^{n} t_i(\theta) - c(p(\theta))\Big].$$

The inequality in the statement of the lemma holds if and only if (p,t) is ex ante budget balanced. □

Before proving Theorem 1 we first reformulate the cartel's problem and then state a version of the separating hyperplane theorem from functional analysis to be used in the proof. Next we introduce some additional notation and prove one preliminary lemma.

From Proposition 1 and Lemma 3 the problem of the cartel facing a grand mechanism Γ with menu (Σ^*, c) and outside options $V_1, \dots, V_n$ (Program C*) can be written as follows:

$$\text{Program C}': \quad \max_{(p,\underline{U}_1,\ldots,\underline{U}_n)} \left\{ \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \underline{U}_i \, d\omega_i(\theta_i) + \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})] \, d\widetilde{\theta}_i \, d\omega_i(\theta_i) \right\},$$

where $p:\Theta \to \Sigma$ and $\underline{U}_i \in \mathbb{R}$ for every i, subject to

F′: $p(\theta) \in \Sigma^*$ for every $\theta \in \Theta$

IC′: $E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$ is non-decreasing in θ_i for every i

IR′: $\underline{U}_i + \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})] \, d\widetilde{\theta}_i - V_i(\theta_i) \geq 0$ for every i and $\theta_i \in [\underline{\theta}, \overline{\theta}]$.

Let Y be a linear vector space and Z a normed linear vector space. Let Z^* be the dual space for Z, that is the space of all bounded linear functionals on Z. Denote by $\langle z, z^* \rangle$ the value of the linear functional $z^* \in Z^*$ at the point $z \in Z$.

Proposition 2 (adapted from Theorem 1, Chapter 8 in Luenberger 1968). *Let Ω be a convex subset of Y. Assume that the positive cone in Z contains an interior point. Let φ be a real-valued concave functional on Ω and G a concave mapping from Ω into Z. Let*

$$\mu_0 = \sup \varphi(y) \text{ subject to } y \in \Omega,\ G(y) \geq \mathbf{0},$$

and assume μ_0 is finite. Then there is an element $z_0^* \geq \mathbf{0}$ in Z^* and $r_0 \geq 0$ such that (r_0, z_0^*) is non-null (either $z_0^* > \mathbf{0}$, or $r_0 > 0$, or both) and

$$\mu_0 = \sup_{y \in \Omega} \{ r_0 \varphi(y) + \langle G(y), z_0^* \rangle \}.$$

Define Ξ to be the space of functions $p:\Theta \to \mathbb{R}^n$ that are integrable for each θ_i. Let $Y = \Xi \times \mathbb{R}^n$. Clearly Y is a linear vector space.

Let Z be the space of n continuous functions $U_i : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$. Adopt a max norm for Z, i.e., for any $(U_1, \ldots, U_n) \in Z$ the norm is defined by $\|(U_1, \ldots, U_n)\| = \max_{t \in [\underline{\theta}, \overline{\theta}], i} |U_i(t)|$. Hence Z is a normed linear vector space. Define the positive cone in Z as the space of n non-negative functions. It is routine to verify that the positive cone in Z has a non-empty interior.[21]

Define the set $\widehat{\Omega} \subset \Xi$ to consist of the functions $p \in \Xi$ satisfying the constraints F', IC', and BB' in Program C′, and let $\Omega = \widehat{\Omega} \times \mathbb{R}^n$. Let the mapping G from Ω into Z be defined by the left-hand side of the inequality constraints IR'. Obviously G is a concave mapping since it is defined by linear functionals on Y. The objective functional φ is linear on Y and thus concave.

Lemma 7. *Ω is convex.*

[21] Section 8.2 in Luenberger (1968).

PROOF. Let $(p, \underline{U}_1, \dots, \underline{U}_n), (p', \underline{U}'_1, \dots, \underline{U}'_n) \in \Omega$. Define

$$(p^\alpha, \underline{U}^\alpha_1, \dots, \underline{U}^\alpha_n) = \alpha(p, \underline{U}_1, \dots, \underline{U}_n) + (1-\alpha)(p', \underline{U}'_1, \dots, \underline{U}'_n) \text{ for } \alpha \in (0,1).$$

We show that $(p^\alpha, \underline{U}^\alpha_1, \dots, \underline{U}^\alpha_n)$ satisfies the constraints F', IC', and BB'. The constraint F' is satisfied since Σ^* is convex:

$$p^\alpha(\theta) = \alpha p(\theta) + (1-\alpha) p'(\theta) \in \Sigma^* \quad \text{for every } \theta.$$

The constraint IC' is satisfied since $E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$ and $E_{\theta_{-i}}[p'_i(\theta_i, \theta_{-i})]$ are non-decreasing:

$$E_{\theta_{-i}}[p^\alpha_i(\theta_i, \theta_{-i})] = \alpha E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})] + (1-\alpha) E_{\theta_{-i}}[p'_i(\theta_i, \theta_{-i})]$$
$$\text{is non-decreasing in } \theta_i \text{ for every } i.$$

The constraint BB' is satisfied because

$$\begin{aligned}
&\sum_{i=1}^n \underline{U}^\alpha_i - E\left[\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p^\alpha_i(\theta) - c(p^\alpha(\theta))\right] \\
&\quad = \alpha\left(\sum_{i=1}^n \underline{U}_i - E\left[\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right]\right) \\
&\qquad + (1-\alpha)\left(\sum_{i=1}^n \underline{U}'_i - E\left[\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p'_i(\theta) - c(p'(\theta))\right]\right) \\
&\qquad + E[c(p^\alpha(\theta)) - \alpha c(p(\theta)) - (1-\alpha) c(p'(\theta))] \\
&\quad \le 0.
\end{aligned}$$

The first two terms are non-positive since $(p, \underline{U}_1, \dots, \underline{U}_n)$ and $(p', \underline{U}'_1, \dots, \underline{U}'_n)$ satisfy the constraint BB'; the last term is non-positive by the convexity of c (see Section 4.1). □

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Assume $(p, \underline{U}_1, \dots, \underline{U}_n)$ solves the problem of the cartel facing a grand mechanism Γ with menu (Σ^*, c), i.e., solves Program C'. Since all the conditions of Proposition 2 are satisfied, there exists $z^*_0 \ge \mathbf{0}$ in the dual space of Z and $r_0 \ge 0$ such that either $z^*_0 > \mathbf{0}$, or $r_0 > 0$, or both. Moreover, by the Riesz representation theorem[22] there exist n bounded non-decreasing right-continuous functions $\Lambda_1, \dots, \Lambda_n$ such that for all $y = (p, \underline{U}_1, \dots, \underline{U}_n) \in Y$ we have

$$\begin{aligned}
\langle G(y), z^*_0\rangle = &\sum_{i=1}^n \int_{\underline{\theta}}^{\overline{\theta}} \underline{U}_i \, d\Lambda_i(\theta_i) \\
&+ \sum_{i=1}^n \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\tilde{\theta}_i, \theta_{-i})] \, d\tilde{\theta}_i\right) d\Lambda_i(\theta_i) - \sum_{i=1}^n \int_{\underline{\theta}}^{\overline{\theta}} V_i(\theta_i) \, d\Lambda_i(\theta_i).
\end{aligned}$$

[22] Theorem 6, Section 36 in Kolmogorov and Fomin (1970).

Define functions $W_1,\ldots,W_n$ as follows: $W_i(\theta_i)=r_0\omega_i(\theta_i)+\Lambda_i(\theta_i)$ for every i and θ_i. Then the value of $(p,\underline{U}_1,\ldots,\underline{U}_n)$ that solves Program C′ also solves

$$\max_{(p,\underline{U}_1,\ldots,\underline{U}_n)}\left\{\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}\underline{U}_i\,dW_i(\theta_i)\right.$$
$$\left.+\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\theta_i}E_{\theta_{-i}}[p_i(\tilde{\theta}_i,\theta_{-i})]\,d\tilde{\theta}_i\,dW_i(\theta_i)-\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}V_i(\theta_i)\,d\Lambda_i(\theta_i)\right\}$$

subject to F', IC', and BB' given in Program C′.

We can drop the last term in the objective function since it is constant. Since (r_0,z_0^*) is non null and $\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}d\omega_i(\theta_i)>0$ we know that

$$\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}dW_i(\theta_i)=r_0\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}d\omega_i(\theta_i)+\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}d\Lambda_i(\theta_i)>0.$$

Hence without loss of generality we can relabel agents in decreasing order of $\int_{\underline{\theta}}^{\overline{\theta}}dW_i(\theta_i)$:

$$\int_{\underline{\theta}}^{\overline{\theta}}dW_1(\theta_1)\geq\int_{\underline{\theta}}^{\overline{\theta}}dW_2(\theta_2)\geq\cdots\geq\int_{\underline{\theta}}^{\overline{\theta}}dW_n(\theta_n)$$

and normalize $\int_{\underline{\theta}}^{\overline{\theta}}dW_1(\theta_1)=1$. If $\int_{\underline{\theta}}^{\overline{\theta}}dW_n(\theta_n)<1$ then simultaneously increasing $\underline{U}_1$ and decreasing $\underline{U}_n$ would allow for an unbounded value for the objective function. Since the solution $(p,\underline{U}_1,\ldots,\underline{U}_n)$ must also solve the original problem, the individual rationality constraints IR' would be violated for agent n, which is a contradiction. Thus $\int_{\underline{\theta}}^{\overline{\theta}}dW_i(\theta_i)=1$ for every i. Without loss of generality we can normalize $W_i(\overline{\theta})=1$ for every i.

Thus we can rewrite the objective function in Program C′ as

$$\sum_{i=1}^{n}\underline{U}_i+\sum_{i=1}^{n}\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\theta_i}E_{\theta_{-i}}[p_i(\tilde{\theta}_i,\theta_{-i})]\,d\tilde{\theta}_i\,dW_i(\theta_i),$$

which proves the result. □

Proof of Corollary 1. According to Theorem 1, $(p,U_1,\ldots,U_n)$ must be cartel interim efficient with respect to the menu (Σ^*,c) relative to the weights $W_1,\ldots,W_n$, i.e., must solve Program CIE. Notice that at the optimum the constraint BB must hold with equality. Hence applying integration by parts we can rewrite the objective function as

$$\begin{aligned}
&\sum_{i=1}^{n} U_i(\underline{\theta}) + \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})]\, d\widetilde{\theta}_i\, dW_i(\theta) \\
&\quad = E\left[\sum_{i=1}^{n} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right] + E\left[\sum_{i=1}^{n} \frac{1-W_i(\theta_i)}{f(\theta_i)} p_i(\theta)\right] \\
&\quad = E\left[\sum_{i=1}^{n} \left(\theta_i + \frac{F(\theta_i)-W_i(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right].
\end{aligned}$$

The constraints F^* coincide with the constraints F, and the constraints IC^* coincide with the first set of constraints IC. If $(p, U_1, \ldots, U_n)$ satisfies BB then it must also satisfy BB^* since

$$0 \le \sum_{i=1}^{n} U_i(\underline{\theta}) \le E\left[\sum_{i=1}^{n} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) p_i(\theta) - c(p(\theta))\right].$$

Hence p is cartel interim efficient with respect to the menu (Σ^*, c) relative to the weights $W_1, \ldots, W_n$, i.e., solves Program CIE*. □

B. Proofs for Section 5

Proof of Theorem 2. Let p be a cartel interim efficient allocation function with respect to the menu (Σ^*, c) relative to the weights $W_1, \ldots, W_n$, i.e., let p solve the Program CIE*.

First we construct a grand mechanism $\Gamma = \langle M, \widetilde{p}, \widetilde{t} \rangle$ with menu (Σ^*, c) that implements the allocation function p. Let the set of messages M_i available to agent i consist of all his possible types $[\underline{\theta}, \overline{\theta}]$ and the rejection message $\varnothing$. If at least one of the agents sends the rejection message $\varnothing$, then nobody receives the good and nobody receives any payment. If none of the agents submits the rejection message then the vector of allocations is $\widetilde{p}(\theta) = p(\theta)$. Define the auxiliary payment function

$$t_i'(\theta) = \theta_i E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})] - \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})]\, d\widetilde{\theta}_i - U_i^N(\underline{\theta}) \quad \text{for every } i \text{ and every } \theta,$$

where

$$U_i^N(\underline{\theta}) = \frac{1}{n} E\left[\sum_{j=1}^{n} \left(\theta_j - \frac{1-F(\theta_j)}{f(\theta_j)}\right) p_j(\theta) - c(p(\theta))\right] \quad \text{for every } i.$$

The interim payoff of agent i of type θ_i is

$$U_i^N(\theta_i) = E_{\theta_{-i}}[\theta_i p_i(\theta_i, \theta_{-i}) - t_i'(\theta_i, \theta_{-i})] = \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i}}[p_i(\widetilde{\theta}_i, \theta_{-i})]\, d\widetilde{\theta}_i + U_i^N(\underline{\theta}),$$

which is at least $U_i^N(\underline{\theta}) \geq 0$ by BB^*. Hence no type of any agent has an incentive to send the rejection message. Using integration by parts,

$$E\left[\sum_{i=1}^{n} t_i'(\theta)\right] = E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)p_i(\theta)\right] - \sum_{i=1}^{n} U_i^N(\underline{\theta})$$
$$= E[c(p(\theta))].$$

According to Lemma 2 there exists a payment function $\widetilde{t} : \Theta \to \mathbb{R}^n$ such that $E_{\theta_{-i}}[\widetilde{t}_i(\theta_i, \theta_{-i})] = E_{\theta_{-i}}[t_i'(\theta_i, \theta_{-i})]$ for every i and $\theta_i \in [\underline{\theta}, \overline{\theta}]$ and $\sum_{i=1}^n \widetilde{t}_i(\theta) = c(p(\theta))$.

By construction, such a grand mechanism Γ has the menu (Σ^*, c). Incentive compatibility is satisfied due to the envelope formula and IC^*. Hence there exists a truth-telling equilibrium in Γ that results in the allocation function p.

Now we show weak collusion-proofness. The solution to the cartel's problem $(\widehat{p}, \widehat{U}_1, \ldots, \widehat{U}_n)$ when facing the grand mechanism Γ with menu (Σ^*, c) and outside options $U_1^N, \ldots, U_n^N$ has to satisfy the individual rationality constraints

$$\widehat{U}_i(\theta_i) \geq U_i^N(\theta_i) \text{ for every } i \text{ and } \theta_i,$$

which implies that the cartel's solution must also satisfy the constraint

$$\sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \widehat{U}_i(\theta_i)\, dW_i(\theta_i) \geq \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i^N(\theta_i)\, dW_i(\theta_i).$$

However p is interim efficient with respect to the menu (Σ^*, c) relative to the weights $W_1, \ldots, W_n$, i.e., solves Program CIE*, and thus p maximizes

$$E\left[\sum_{i=1}^{n}\left(\theta_i + \frac{F(\theta_i) - W_i(\theta_i)}{f(\theta_i)}\right)p_i(\theta) - c(p(\theta))\right] = \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i^N(\theta_i)\, dW_i(\theta_i).$$

Hence

$$\sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \widehat{U}_i(\theta_i)\, dW_i(\theta_i) = \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i^N(\theta_i)\, dW_i(\theta_i),$$

and thus the allocation function $\widehat{p}$ must also be cartel interim efficient with respect to the menu (Σ^*, c) relative to the weights $W_1, \ldots, W_n$, i.e., it must solve Program CIE*. By the essential uniqueness condition we also must have $\widehat{U}_i(\theta_i) = U_i^N(\theta_i)$ for every θ_i and i, which implies

$$\sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} \widehat{U}_i(\theta_i)\, d\omega_i(\theta_i) = \sum_{i=1}^{n} \int_{\underline{\theta}}^{\overline{\theta}} U_i^N(\theta_i)\, d\omega_i(\theta_i).$$

Hence $(p, U_1^N, \ldots, U_n^N)$ solves the cartel's problem (Program C*), which gives the result. □

PROOF OF THEOREM 3. Assume $\theta^* + (F(\theta^*)/f(\theta^*)) \geq \Pi^*/(1 - F^n(\theta^*))$. Denote $R^* = \Pi^*/(1-F^n(\theta^*))$. Consider the linear and symmetric menu

$$\Sigma^* = \Sigma \quad \text{and} \quad c(p) = R^* \sum_{i=1}^{n} p_i \text{ for every } p \in \Sigma.$$

Consider the symmetric weights

$$W_i(\theta_i) = \overline{W}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \in [\underline{\theta}, \overline{\theta}) \\ 1 & \text{if } \theta_i = \overline{\theta} \end{cases} \quad \text{for every } i.$$

We show that a mechanism that implements an interim efficient allocation with respect to such a menu relative to such weights achieves the Myerson revenue. Consider the problem describing this particular cartel interim efficient allocation (Program CIE):

$$\max_p E\left[\sum_{i=1}^{n}\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)} - R^*\right) p_i(\theta_1, \ldots, \theta_n)\right] \quad \text{subject to}$$

F^*: $p(\theta_1, \ldots, \theta_n) \in \Sigma$ for every $(\theta_1, \ldots, \theta_n) \in \Theta$

IC^*: $E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$ is non-decreasing in θ_i for every i

$$BB^*\!: \; E\left[\sum_{i=1}^{n}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - R^*\right) p_i(\theta_1, \ldots, \theta_n)\right] \geq 0.$$

The Lagrangian for this problem is

$$E\left[\sum_{i=1}^{n}\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)} - R^* + \lambda\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - R^*\right)\right) p_i(\theta_1, \ldots, \theta_n)\right]$$
$$= (1+\lambda) E\left[\sum_{i=1}^{n}\left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)} - \frac{\lambda}{1+\lambda}\frac{1}{f(\theta_i)} - R^*\right) p_i(\theta_1, \ldots, \theta_n)\right],$$

where $\lambda \geq 0$. Notice that $\theta_i + (F(\theta_i)/f(\theta_i)) - (\lambda/(1+\lambda))(1/f(\theta_i)) - R^*$ is increasing for every $\lambda \geq 0$ by Condition 1. Hence the solution must have the form

$$p_i(\theta_1, \ldots, \theta_n) = \begin{cases} 1 & \text{if } \theta_i > \max\{\max_{j \neq i} \theta_j, \widehat{\theta}\} \\ 0 & \text{otherwise} \end{cases} \quad \text{for every } i \text{ and } (\theta_1, \ldots, \theta_n),$$

where $\widehat{\theta}$ is such that $\widehat{\theta} + (F(\widehat{\theta})/f(\widehat{\theta})) - (\lambda/(1+\lambda))(1/f(\widehat{\theta})) = R^*$ if such a value of $\widehat{\theta}$ exists and $\overline{\theta}$ otherwise. Notice that the constraint IC^* is automatically satisfied.

Assume the constraint BB^* does not bind. Then $\lambda = 0$ and $\widehat{\theta} + (F(\widehat{\theta})/f(\widehat{\theta})) = R^*$. Such a value of $\widehat{\theta}$ exists since $R^* \in (\underline{\theta}, \overline{\theta})$. Recall that $\theta^* + (F(\theta^*)/f(\theta^*)) \geq R^*$, which by Condition 1 implies $\widehat{\theta} \leq \theta^*$.

Under the allocation p the constraint BB^* becomes

$$\int_{\widehat{\theta}}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)} - R^*\right) dF^n(\theta) \geq 0.$$

Also notice that

$$\int_{\theta^*}^{\overline{\theta}} \left(\theta - \frac{1-F(\theta)}{f(\theta)} - R^*\right) dF^n(\theta) = \Pi^* - \frac{\Pi^*}{1-F^n(\theta^*)}(1-F^n(\theta^*)) = 0.$$

However, by definition of θ^*,

$$\theta - \frac{1-F(\theta)}{f(\theta)} - R^* < \theta^* - \frac{1-F(\theta^*)}{f(\theta^*)} - R^* = -R^* < 0 \quad \text{for every } \theta \in (\widehat{\theta}, \theta^*),$$

which gives a contradiction. Hence the constraint BB^* does bind and $\widehat{\theta} = \theta^*$. The revenue of the principal is equal to the Myerson revenue:

$$R^* E\left[\sum_{i=1}^n p_i(\theta_1, \ldots, \theta_n)\right] = \frac{\Pi^*}{1-F^n(\theta^*)}(1-F^n(\theta^*)) = \Pi^*.$$

Now assume the Myerson revenue can be achieved by a mechanism that implements an interim efficient allocation with respect to some menu (Σ^*, c) relative to some weights $W_1, \ldots, W_n$. The Revenue Equivalence Theorem[23] implies that to achieve the Myerson revenue Π^* this mechanism must implement the Myerson allocation. Hence

$$\begin{aligned} E[c(p(\theta_1, \ldots, \theta_n))] &= \sum_{i=1}^n \frac{1}{n}(1-F^n(\theta^*))c(e^i) + F^n(\theta^*)c(\mathbf{0}) \\ &= \left(\frac{1}{n}\sum_{i=1}^n c(e^i) - c(\mathbf{0})\right)(1-F^n(\theta^*)) + c(\mathbf{0}) = \Pi^*, \end{aligned}$$

where $e^i \in \mathbb{R}^n$ and $e^i_i = 1$, $e^i_j = 0$ for $j \neq i$.

Since $c(\mathbf{0}) \leq 0$ we have

$$\frac{1}{n}\sum_{i=1}^n (c(e^i) - c(\mathbf{0})) \geq \frac{\Pi^*}{1-F^n(\theta^*)}.$$

By the optimality of the allocation p we have

$$\theta^* + \frac{F(\theta^*) - W_i(\theta^*)}{f(\theta^*)} \geq c(e^i) - c(\mathbf{0}) \text{ for every } i.$$

Notice that

$$\theta^* + \frac{F(\theta^*)}{f(\theta^*)} \geq \frac{1}{n}\sum_{i=1}^n \left(\theta^* + \frac{F(\theta^*) - W_i(\theta^*)}{f(\theta^*)}\right) \geq \frac{1}{n}\sum_{i=1}^n (c(e^i) - c(\mathbf{0})) \geq \frac{\Pi^*}{1-F^n(\theta^*)},$$

which gives the result. □

[23] See Myerson (1981).

Proof of Lemma 5. Let $\widehat{p}$ be an interim efficient allocation with respect to the linear and symmetric menu with price R relative to the weight functions $\overline{W}, \ldots, \overline{W}$. The form of the solution $\widehat{p}$ is derived in Theorem 3. Denote the principal's revenue by $\widehat{\pi}$ and note that

$$\widehat{\pi} = RE\left[\sum_{i=1}^{n} \widehat{p}_i(\theta_1, \ldots, \theta_n)\right] = R(1 - F^n(\hat{\theta})),$$

where $\hat{\theta}$ is the cutoff type.

Let p' be an interim efficient allocation with respect to the linear and symmetric menu with price R relative to the weight functions $W_1, \ldots, W_n$. Denote $\theta_i' = \inf\{\theta_i : E_{\theta_{-i}}[p_i'(\theta_i, \theta_{-i})] > 0\}$ and $k = \arg\min_i \theta_i'$. Denote the principal's revenue by π' and assume that $\pi' > \widehat{\pi}$.

Case 1 The allocation $\widehat{p}$ is such that the constraint BB^* does not bind.

By the optimality of the allocation $\widehat{p}$ we have $\widehat{\theta} + (F(\widehat{\theta})/f(\widehat{\theta})) = R$, and by the optimality of the allocation p' we have $\theta_k' + ((F(\theta_k') - W_k(\theta_k'))/f(\theta_k')) \geq R$. This implies that $\widehat{\theta} \leq \theta_k'$.

By the definition of θ_k' we must have

$$E\left[\sum_{i=1}^{n} p_i'(\theta_1, \ldots, \theta_n) \mid \max_i \theta_i < \theta_k'\right] = 0.$$

Hence

$$\begin{aligned}\pi' &= RE\left[\sum_{i=1}^{n} p_i'(\theta_1, \ldots, \theta_n) \mid \max_i \theta_i > \theta_k'\right] \Pr\{\max_i \theta_i > \theta_k'\} \\ &\leq R(1 - F^n(\theta_k')) \leq R(1 - F^n(\widehat{\theta})) = \widehat{\pi},\end{aligned}$$

where the first inequality follows from the facts $\sum_{i=1}^{n} p_i'(\theta) \leq 1$ and $\Pr\{\max_i \theta_i > \theta_k'\} = (1 - F^n(\theta_k'))$, and the second inequality follows from $\widehat{\theta} \leq \theta_k'$. This contradicts our assumption that $\pi' > \widehat{\pi}$.

Case 2 The allocation $\widehat{p}$ is such that the constraint BB^* binds.

Consider the following auxiliary problem.

$$\max_p E\left[\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - R\right) p_i(\theta_1, \ldots, \theta_n)\right] \quad \text{subject to}$$

$$F^*: \; p(\theta_1, \ldots, \theta_n) \in \Sigma \text{ for every } (\theta_1, \ldots, \theta_n) \in \Theta$$

$$IC^*: \; E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})] \text{ is non-decreasing in } \theta_i \text{ for every } i$$

$$A^*: \; RE\left[\sum_{i=1}^{n} p_i(\theta_1, \ldots, \theta_n)\right] = \pi'.$$

The Lagrangian for this problem is

$$E\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}-(1-\lambda)R\right)p_i(\theta_1,\dots,\theta_n)\right].$$

Notice that $\theta_i-((1-F(\theta_i))/f(\theta_i))-(1-\lambda)R$ is increasing by Condition 1. Hence the solution must take the form

$$p''_i(\theta_1,\dots,\theta_n)=\begin{cases}1 & \text{if } \theta_i>\max\{\max_{j\neq i}\theta_j,\theta''\}\\ 0 & \text{otherwise}\end{cases}\quad \text{for every } i \text{ and } (\theta_1,\dots,\theta_n),$$

where θ'' is such that $\theta''-((1-F(\theta''))/f(\theta''))=(1-\lambda)R$ if such a value of θ'' exists and $\overline{\theta}$ otherwise. By construction we have

$$\pi'=RE\left[\sum_{i=1}^{n}p''_i(\theta_1,\dots,\theta_n)\right]=R(1-F^n(\theta'')).$$

Notice that our assumption that $\pi'>\widehat{\pi}$ implies that $\theta''<\widehat{\theta}$. Then

$$\begin{aligned}0&=E\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}-R\right)\widehat{p}_i(\theta_1,\dots,\theta_n)\right]=\int_{\widehat{\theta}}^{\overline{\theta}}(\theta-\frac{1-F(\theta)}{f(\theta)}-R)\,dF^n(\theta)\\&>\int_{\theta''}^{\overline{\theta}}\left(\theta-\frac{1-F(\theta)}{f(\theta)}-R\right)dF^n(\theta)=E\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}-R\right)p''_i(\theta_1,\dots,\theta_n)\right]\\&\geq E\left[\sum_{i=1}^{n}\left(\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}-R\right)p'_i(\theta_1,\dots,\theta_n)\right],\end{aligned}$$

where the first equality is due to the fact that the constraint BB^* binds at $\widehat{p}$, the second equality utilizes the form of $\widehat{p}$, the first inequality uses the fact that $\theta-((1-F(\theta))/f(\theta))-R<0$ for every $\theta\leq\widehat{\theta}$, the third equality utilizes the form of p'', and the final inequality holds since p'' solves the auxiliary problem. We have reached a contradiction since p' violates the constraint BB^*. □

Before proving Theorem 4 we establish a technical result that helps to find the price that maximizes the revenue of the principal.

Lemma 8. *There exists a unique $\theta'\in(\underline{\theta},\overline{\theta})$ such that*

$$\int_{\theta}^{\overline{\theta}}\left(\widetilde{\theta}-\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}-\left(\theta+\frac{F(\theta)}{f(\theta)}\right)\right)dF^n(\widetilde{\theta})\gtreqqless 0 \text{ when } \theta\lesseqqgtr\theta'.$$

Proof. Define

$$R(\theta)=\frac{1}{1-F^n(\theta)}\int_{\theta}^{\overline{\theta}}\left(\widetilde{\theta}-\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}\right)dF^n(\widetilde{\theta})\text{ for }\theta\in[\underline{\theta},\overline{\theta}),$$

and

$$R(\overline{\theta}) = \lim_{\theta\to\overline{\theta}} R(\theta).$$

We need to show that there exists a unique $\theta' \in (\underline{\theta}, \overline{\theta})$ such that

$$R(\theta) \gtreqless \theta + \frac{F(\theta)}{f(\theta)} \text{ when } \theta \lesseqgtr \theta'.$$

Assume for a moment that we have established the following properties: (i) $R(\underline{\theta}) > \underline{\theta}$; (ii) $R(\overline{\theta}) = \overline{\theta}$; and (iii) $R'(\theta) < 1$ for every $\theta < \overline{\theta}$.

Notice that by Condition 1,

$$\frac{d}{d\theta}\left(\theta + \frac{F(\theta)}{f(\theta)}\right) \geq 1 \quad \text{for every } \theta.$$

Thus we have

$$\underline{\theta} + \frac{F(\underline{\theta})}{f(\underline{\theta})} = \underline{\theta} < R(\underline{\theta}),\ \overline{\theta} + \frac{F(\overline{\theta})}{f(\overline{\theta})} > \overline{\theta} = R(\overline{\theta}) \quad \text{and} \quad \frac{d}{d\theta}\left(\theta + \frac{F(\theta)}{f(\theta)}\right) \geq 1 > R'(\theta).$$

This implies that there exists a unique $\theta' \in (\underline{\theta}, \overline{\theta})$ with the desired properties.

In the remainder of the proof we establish (i), (ii), and (iii). Property (i) holds because $R(\underline{\theta})$ is just the expectation of the second-order statistic:

$$R(\underline{\theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \left(\tilde{\theta} - \frac{1 - F(\tilde{\theta})}{f(\tilde{\theta})}\right) dF^n(\tilde{\theta}) > \underline{\theta}.$$

Property (ii) holds because

$$R(\overline{\theta}) = \lim_{\theta\to\overline{\theta}} \frac{\int_{\theta}^{\overline{\theta}} \left(\tilde{\theta} - \frac{1 - F(\tilde{\theta})}{f(\tilde{\theta})}\right) dF^n(\tilde{\theta})}{1 - F^n(\theta)} = \lim_{\theta\to\overline{\theta}} \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) = \overline{\theta}.$$

To establish property (iii) we first use integration by parts to show that

$$R(\theta) - \theta = \frac{1}{1 - F^n(\theta)} \int_{\theta}^{\overline{\theta}} (1 - F^n(\tilde{\theta}) - nF^{n-1}(\tilde{\theta})(1 - F(\tilde{\theta})))\, d\tilde{\theta}.$$

Differentiation and the fact that

$$1 - F^n(\theta) = \int_{\theta}^{\overline{\theta}} nf(\tilde{\theta})F^{n-1}(\tilde{\theta})\, d\tilde{\theta}$$

give

$$R'(\theta) - 1 = \frac{nf(\theta)F^{n-1}(\theta)}{(1 - F^n(\theta))^2}\left(\int_{\theta}^{\overline{\theta}} (1 - F^n(\tilde{\theta}) - nF^{n-1}(\tilde{\theta})(1 - F(\tilde{\theta})))\, d\tilde{\theta}\right.$$
$$\left. - \int_{\theta}^{\overline{\theta}} nf(\tilde{\theta})F^{n-1}(\tilde{\theta})\, d\tilde{\theta} \cdot \frac{(1 - F^n(\theta) - nF^{n-1}(\theta)(1 - F(\theta)))}{nf(\theta)F^{n-1}(\theta)}\right).$$

A sufficient condition for this expression to be negative is

$$1-F^n(\widetilde{\theta})-nF^{n-1}(\widetilde{\theta})(1-F(\widetilde{\theta}))<nf(\widetilde{\theta})F^{n-1}(\widetilde{\theta})\left(\frac{1-F^n(\theta)}{nf(\theta)F^{n-1}(\theta)}-\frac{1-F(\theta)}{f(\theta)}\right)$$

for every $\theta,\widetilde{\theta}$ such that $\theta<\widetilde{\theta}<\overline{\theta}$. After rearranging we get

$$\left(\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}\right)\left(\frac{1+F(\widetilde{\theta})+\cdots+F^{n-1}(\widetilde{\theta})}{nF^{n-1}(\widetilde{\theta})}-1\right)<\left(\frac{1-F(\theta)}{f(\theta)}\right)\left(\frac{1+F(\theta)+\cdots+F^{n-1}(\theta)}{nF^{n-1}(\theta)}-1\right),$$

which is confirmed since by Condition 1 we have

$$\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}\leq\frac{1-F(\theta)}{f(\theta)},$$

and since $F(\widetilde{\theta})>F(\theta)$ implies

$$\frac{1+F(\widetilde{\theta})+\cdots+F^{n-1}(\widetilde{\theta})}{nF^{n-1}(\widetilde{\theta})}-1<\frac{1+F(\theta)+\cdots+F^{n-1}(\theta)}{nF^{n-1}(\theta)}-1. \qquad \square$$

Proof of Theorem 4. Let $\widehat{p}$ be an interim efficient allocation with respect to the linear and symmetric menu with price R relative to the weight functions $\overline{W},\dots,\overline{W}$. The form of the solution $\widehat{p}$ is derived in Theorem 3.

First we show that $\theta^*<\theta'$. If $R=\theta^*+(F(\theta^*)/f(\theta^*))$ then the cartel's optimal cutoff is θ^*. The left-hand side of the constraint BB^* is

$$\int_{\theta^*}^{\overline{\theta}}\left(\widetilde{\theta}-\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}-\left(\theta^*+\frac{F(\theta^*)}{f(\theta^*)}\right)\right)dF^n(\widetilde{\theta})=\Pi^*-\left(\theta^*+\frac{F(\theta^*)}{f(\theta^*)}\right)(1-F^n(\theta^*))>0,$$

where the inequality follows from the assumption that $\theta^*+(F(\theta^*)/f(\theta^*))<\Pi^*/(1-F^n(\theta^*))$. Thus by Lemma 8 we have $\theta^*<\theta'$.

Notice that

$$\frac{d}{d\theta}\left[\int_{\theta}^{\overline{\theta}}\left(\widetilde{\theta}-\frac{1-F(\widetilde{\theta})}{f(\widetilde{\theta})}\right)dF^n(\widetilde{\theta})\right]=-\left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)nf(\theta)F^{n-1}(\theta)\gtreqless 0 \text{ when } \theta\lesseqgtr\theta^*.$$

Thus $\int_{\theta}^{\overline{\theta}}(\widetilde{\theta}-((1-F(\widetilde{\theta}))/f(\widetilde{\theta})))\,dF^n(\widetilde{\theta})$ is decreasing on the interval $[\theta',\overline{\theta}]$.

Consider $R<\theta'+(F(\theta')/f(\theta'))$. Then the cartel's optimal cutoff is $\widehat{\theta}$ such that $\widehat{\theta}+(F(\widehat{\theta})/f(\widehat{\theta}))=R$, which implies that $\widehat{\theta}<\theta'$. By Lemma 8, BB^* does not bind and the revenue is

$$R(1-F^n(\widehat{\theta}))=\left(\widehat{\theta}+\frac{F(\widehat{\theta})}{f(\widehat{\theta})}\right)(1-F^n(\widehat{\theta}))\leq\left(\theta^{**}+\frac{F(\theta^{**})}{f(\theta^{**})}\right)(1-F^n(\theta^{**})),$$

where the inequality comes from the fact that θ^{**} is a maximizer of $(\theta+(F(\theta)/f(\theta)))\times(1-F^n(\theta))$ over $[\underline{\theta},\theta']$.

Consider $R \geq \theta' + (F(\theta')/f(\theta'))$. Then the cartel's optimal cutoff is $\widehat{\theta}$ such that $\widehat{\theta} + (F(\widehat{\theta})/f(\widehat{\theta})) = R$, which implies that $\widehat{\theta} \geq \theta'$. By Lemma 8, BB^* is violated unless $\widehat{\theta} = \theta'$, and hence the implemented cutoff θ^0 is

$$\theta^0 = \min_{\theta} \left\{ \theta : \int_{\theta}^{\overline{\theta}} \left(\widetilde{\theta} - \frac{1 - F(\widetilde{\theta})}{f(\widetilde{\theta})} - R \right) dF^n(\widetilde{\theta}) \right\}.$$

By Condition 1 we have $\theta^0 \geq \widehat{\theta}$, and thus $\theta^0 \geq \theta'$. The revenue is

$$\begin{aligned} R(1 - F^n(\theta^0)) &= \int_{\theta^0}^{\overline{\theta}} \left(\widetilde{\theta} - \frac{1 - F(\widetilde{\theta})}{f(\widetilde{\theta})} \right) dF^n(\widetilde{\theta}) \leq \int_{\theta'}^{\overline{\theta}} \left(\widetilde{\theta} - \frac{1 - F(\widetilde{\theta})}{f(\widetilde{\theta})} \right) dF^n(\widetilde{\theta}) \\ &= \left(\theta' + \frac{F(\theta')}{f(\theta')} \right) (1 - F^n(\theta')) \leq \left(\theta^{**} + \frac{F(\theta^{**})}{f(\theta^{**})} \right) (1 - F^n(\theta^{**})), \end{aligned}$$

where the first equality follows from the binding constraint BB^*, the first inequality comes from the fact that $\int_{\theta}^{\overline{\theta}} (\widetilde{\theta} - ((1 - F(\widetilde{\theta}))/f(\widetilde{\theta}))) \, dF^n(\widetilde{\theta})$ is decreasing on the interval $[\theta', \overline{\theta}]$, the second equality is by the definition of θ', and the last inequality comes from the fact that θ^{**} is a maximizer of $(\theta + (F(\theta)/f(\theta)))(1 - F^n(\theta))$ over $[\underline{\theta}, \theta']$. □

C. Proofs for Section 6

Proof of Lemma 6. Assume (p, t) is feasible and budget balanced in Γ. Take any $\theta \in \Theta$. Since $p(\theta) = E_{\phi(\theta)}[\widetilde{p}(m) + z(m, \theta)]$ and $\widetilde{p}(m) + z(m, \theta) \in \Sigma_1$ for every $m \in M$ and $\theta \in \Theta$, we have $p(\theta) \in \mathrm{co}(\Sigma_1^r) = \Sigma^r$. By the definition of the function c^r we must have

$$\sum_{i=1}^{n} \widetilde{t}_i(m) \geq c^r(\widetilde{p}(m) + z(m, \theta)) \text{ for every } m \in M \text{ and } \theta \in \Theta.$$

Hence

$$E_{\phi(\theta)}\left[\sum_{i=1}^{n} \widetilde{t}_i(m) \right] \geq E_{\phi(\theta)}[c^r(\widetilde{p}(m) + z(m, \theta))] \geq c^r(E_{\phi(\theta)}[\widetilde{p}(m) + z(m, \theta)]) = c^r(p(\theta)),$$

where the second inequality follows from the convexity of c^r. Using the budget balance condition,

$$0 \geq \sum_{i=1}^{n} y_i(\theta) = E_{\phi(\theta)}\left[\sum_{i=1}^{n} \widetilde{t}_i(m) \right] - \sum_{i=1}^{n} t_i(\theta) \geq c^r(p(\theta)) - \sum_{i=1}^{n} t_i(\theta).$$

Hence $p(\theta)$ and $t(\theta)$ satisfy (i) and (ii).

Take any $\theta \in \Theta$ and consider a vector of allocations $p(\theta)$ and a vector of payments $t(\theta)$ that satisfy (i) and (ii).

Define a set $D_1^r = \{(\sum_{i=1}^n \widetilde{p}_i(m), \sum_{i=1}^n \widetilde{t}_i(m)) \mid m \in M\}$, and let $D^r = \mathrm{co}(D_1^r)$. Notice that $(\sum_{i=1}^n p_i(\theta), c^r(p(\theta))) \in D^r$ by the definition of c^r and Condition 2. By

Carathéodory's theorem, any point in the set D^r can be represented as a convex combination of at most three points in the set D_1^r. Thus there exist $d^1,\dots,d^K$ in the set D_1^r (where $K \le 3$) and $\sigma_1,\dots,\sigma_K$ from $\mathbb{R}_+^K$ such that $\sum_{k=1}^K \sigma_k = 1$ and

$$\left(\sum_{i=1}^n p_i(\theta), c(p(\theta))\right) = \sum_{k=1}^K \sigma_k d^k.$$

By the definition of the set D_1^r there exist messages $m^1,\dots,m^K$ such that

$$\left(\sum_{i=1}^n \widetilde{p}(m^k), \sum_{i=1}^n \widetilde{t}_i(m^k)\right) = d^k \quad \text{for every } k.$$

Hence the cartel can achieve an allocation $\sum_{k=1}^K \sigma_k \widetilde{p}(m^k)$ and a vector of payments $\sum_{k=1}^K \sigma_k \widetilde{t}(m^k)$ by sending the messages $m^1,\dots,m^K$ with the probabilities $\sigma_1,\dots,\sigma_K$.

To achieve a desired allocation consider the following reallocation adjustments:

$$z(m^k,\theta) = p(\theta)\left(\frac{\sum_{i=1}^n \widetilde{p}(m^k)}{\sum_{i=1}^n p_i(\theta)}\right) - \widetilde{p}(m^k) \quad \text{for every } k.$$

They are feasible since $\widetilde{p}(m^k) + z(m^k,\theta)$ has only nonnegative entries and

$$\sum_{i=1}^n (\widetilde{p}_i(m^k) + z_i(m^k,\theta)) = \sum_{i=1}^n \widetilde{p}_i(m^k) \le 1 \quad \text{for every } k.$$

We also need to verify that $z(m^k,\theta)$ is balanced for every k:

$$\sum_{i=1}^n z_i(m^k,\theta) = \sum_{i=1}^n p_i(\theta)\left(\frac{\sum_{i=1}^n \widetilde{p}(m^k)}{\sum_{i=1}^n p_i(\theta)}\right) - \sum_{i=1}^n \widetilde{p}_i(m^k) = 0.$$

Let us verify that these reallocation adjustments result in the desired allocation:

$$\sum_{k=1}^K \sigma_k(\widetilde{p}(m^k) + z(m^k,\theta)) = p(\theta)\left(\frac{\sum_{k=1}^K \sigma_k \sum_{i=1}^n \widetilde{p}(m^k)}{\sum_{i=1}^n p_i(\theta)}\right) = p(\theta).$$

To achieve the desired vector of payments $t(\theta)$ we need the following vector of side payments:

$$y(\theta) = \sum_{k=1}^K \sigma_k \widetilde{t}(m^k) - t(\theta).$$

Finally, we need to verify that $y(\theta)$ is budget balanced:

$$\sum_{i=1}^n y_i(\theta) = \sum_{i=1}^n \sum_{k=1}^K \sigma_k \widetilde{t}_i(m^k) - \sum_{i=1}^n t_i(\theta) = c^r(p(\theta)) - \sum_{i=1}^n t_i(\theta) \le 0,$$

where the inequality follows from (ii). Hence $(p(\theta), t(\theta))$ is feasible and budget balanced in Γ. □

Proof of Theorem 5. We give the proof for the case $\theta^*+(F(\theta^*)/f(\theta^*))\geq\Pi^*/(1-F^n(\theta^*))$. The argument for the other case is similar.

Consider the weakly collusion-proof grand mechanism Γ from Theorem 2 that implements the cartel interim efficient allocation p. Modify this grand mechanism by extending the set of messages M_i available to agent i with the following subset of messages: "buy the good now at the price R'", where R' can be chosen from the interval

$$\left(\overline{\theta}-\int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\,d\theta,\infty\right).$$

If none of the agents chooses "buy the good now" then the game proceeds as before. If one of the agents chooses "buy the good now", then he gets the good for sure at the price R', while the other agents get zero payoffs, and the game is over. If several agents choose "buy the good now", then the agent with the lowest price R' (and the lowest index i in case several agents chose the same price R') gets the good for sure at this price, while all the other agents get zero payoffs, and the game is over.

The interim payoff of agent i of type θ_i from the truth-telling equilibrium in the mechanism Γ equilibrium is given by

$$U_i^N(\theta_i)=\max\left\{0,\int_{\theta^*}^{\theta_i} F^{n-1}(\theta)\,d\theta\right\}.$$

The payoff of agent i of type θ_i from sending the message "buy the good now" is at most

$$\begin{aligned}\theta_i-\left(\overline{\theta}-\int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\,d\theta\right)&=\int_{\theta^*}^{\theta_i} F^{n-1}(\theta)\,d\theta-\int_{\theta_i}^{\overline{\theta}}(1-F^{n-1}(\theta))\,d\theta\\ &\leq\int_{\theta^*}^{\theta_i} F^{n-1}(\theta)\,d\theta\leq U_i^N(\theta_i).\end{aligned}$$

Hence the truth-telling equilibrium in the mechanism Γ remains an equilibrium.

Next we show that the modification of the grand mechanism does not change the menu the cartel is facing. Notice that

$$\left(\overline{\theta}-\int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\,d\theta\right)(1-F^n(\theta^*))=\int_{\theta^*}^{\overline{\theta}}\left(\overline{\theta}-\int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\,d\theta\right)dF^n(\theta),$$

and

$$\Pi^*=\int_{\theta^*}^{\overline{\theta}}\left(\theta-\frac{1-F(\theta)}{f(\theta)}\right)dF^n(\theta)=\int_{\theta^*}^{\overline{\theta}}\left(\theta-\frac{\int_{\theta^*}^{\theta} F^{n-1}(\tilde{\theta})\,d\tilde{\theta}}{F^{n-1}(\theta)}\right)dF^n(\theta).$$

Since

$$\theta-\frac{\int_{\theta^*}^{\theta} F^{n-1}(\tilde{\theta})\,d\tilde{\theta}}{F^{n-1}(\theta)}$$

is strictly increasing on $[\theta^*, \overline{\theta}]$ we have

$$\overline{\theta} - \int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\, d\theta \geq R = \frac{\Pi^*}{1 - F^n(\theta^*)}.$$

Hence the menu of the grand mechanism Γ remains linear and symmetric with the price $R = \Pi^*/(1 - F^n(\theta^*))$.

Now we show unique collusion-proofness. Consider any equilibrium of the game, which may have either acceptance or rejection of the collusive proposal on the equilibrium path. Denote by $\widehat{p}$ the allocation function and by $(\widehat{U}_1, \ldots, \widehat{U}_n)$ the interim payoffs resulting from this equilibrium. If $(\widehat{p}, \widehat{U}_1, \ldots, \widehat{U}_n) \neq (p, U_1^N, \ldots, U_n^N)$, then $(\widehat{p}, \widehat{U}_1, \ldots, \widehat{U}_n)$ is not cartel interim efficient with respect to the linear symmetric menu with a price R relative to the weight functions $\overline{W}, \ldots, \overline{W}$. On the other hand, $(p, U_1^N, \ldots, U_n^N)$ is cartel interim efficient with respect to the linear symmetric menu with the price R relative to the weight functions $\overline{W}, \ldots, \overline{W}$, and, moreover, the allocation function p is essentially unique. Since these weight functions assign all the welfare weight to the highest type $\overline{\theta}$ we have

$$\sum_{i=1}^{n} \widehat{U}_i(\overline{\theta}) < \sum_{i=1}^{n} U_i^N(\overline{\theta}).$$

Thus there exists an agent i such that $\widehat{U}_i(\overline{\theta}) < U_i^N(\overline{\theta})$. However agent i of type $\overline{\theta}$ can alternatively choose to "buy the good now at the price $\overline{\theta} - \int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\, d\theta + \varepsilon$" to secure the payoff

$$\overline{\theta} - \left(\overline{\theta} - \int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\, d\theta + \varepsilon \right) = \int_{\theta^*}^{\overline{\theta}} F^{n-1}(\theta)\, d\theta - \varepsilon = U_i^N(\overline{\theta}) - \varepsilon.$$

For sufficiently small ε this payoff is greater than $\widehat{U}_i(\overline{\theta})$. Hence there cannot be such an equilibrium, and thus every equilibrium of the game results in the allocation function p and interim payoffs $(U_1^N, \ldots, U_n^N)$. □

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